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Review

SLE for theoretical physicists

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Abstract

This article provides an introduction to Schramm (stochastic)–Loewner evolution (SLE) and to its connection with conformal field theory, from the point of view of its application to two-dimensional critical behaviour. The emphasis is on the conceptual ideas rather than rigorous proofs.

1. Introduction

1.1. Historical overview

The study of critical phenomena has long been a breeding ground for new ideas in theoretical physics. Such behaviour is characterised by a diverging correlation length and cannot easily be approximated by considering small systems with only a few degrees of freedom. Initially, it appeared that the analytic study of such problems was a hopeless task, although self-consistent approaches such as mean field theory were often successful in providing a semi-quantitative description.

Following Onsager's calculation of the free energy of the square lattice Ising model in 1944, steady progress was made in the exact solution of an ever-increasing

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number of lattice models in two dimensions. While many of these are physically relevant, and the techniques used have spawned numerous spin-offs in the theory of integrable systems, it is fair to say that these methods have not cast much light on the general nature of the critical state. In addition, virtually no progress has been made from this direction in finding analytic solutions to such simple and important problems as percolation.

An important breakthrough occurred in the late 1960's, with the development of renormalisation group (RG) ideas by Wilson and others. The fundamental realisation was that, in the scaling limit where both the correlation length and all other macroscopic length scales are much larger than that of the microscopic interactions, classical critical systems are equivalent to renormalisable quantum field theories in euclidean space-time. Since there is often only a finite or a denumerable set of such field theories with given symmetries, at a stroke this explained the observed phenomenon of universality: systems with very different constituents and microscopic interactions nevertheless exhibit the same critical behaviour in the scaling limit. This single idea has led to a remarkable unification of the theoretical bases of particle physics, statistical mechanics and condensed matter theory, and has led to extensive cross-fertilisation between these disciplines. These days, a typical paper using the ideas and methods of quantum field theory is as likely to appear in a condensed matter physics journal as in a particle physics publication (although there seems to be a considerable degree of conservatism among the writers of field theory text books in recognising this fact).

Two important examples of this interdisciplinary flow were the development of lattice gauge theories in particle physics, and the application of conformal field theory (CFT), first developed as a tool in string theory, to statistical mechanics and condensed matter physics. As will be explained later, in two-dimensional classical systems and quantum systems in $1+1$ dimensions conformal symmetry is extremely powerful, and has led to a cornucopia of new exact results. Essentially, the RG programme of classifying all suitable renormalisable quantum field theories in two dimensions has been carried through to its conclusion in many cases, providing exact expressions for critical exponents, correlation functions, and other universal quantities. However the geometrical, as opposed to the algebraic, aspects of conformal symmetry are not apparent in this approach.

One minor but nevertheless theoretically influential prediction of these methods was the conjectured crossing formula [1] for the probability that, in critical percolation, a cluster should exist which spans between two disjoint segments of the boundary of some simply connected region (a more detailed account of this problem will be given later). With this result, the simmering unease that mathematicians felt about these methods came to the surface (see, for example, the comments in [2]). What exactly are these renormalised local operators whose correlation functions the field theorists so happily manipulate, according to rules that sometimes seem to be a matter of cultural convention rather than any rigorous logic? What does conformal symmetry really mean? Exactly which object is conformally invariant? And so on. Aside from these deep concerns, there was perhaps also the territorial feeling that percolation theory, in particular, is a branch of probability theory, and should be understood from that point of view, not merely as a by-product of quantum field theory.

Thus, it was that a number of pure mathematicians, versed in the methods of probability theory, stochastic analysis and conformal mapping theory, attacked this problem. Instead of trying make rigorous the notions of field theory about local operators, they focused on the random curves which form the boundaries of clusters on the lattice, and on what should be the properties of the measure on such curves in the continuum limit as the lattice spacing approaches zero. The idea of thinking about lattice models this way was not new: in particular in the 1980s it led to the very successful but non-rigorous Coulomb gas approach [3] to two-dimensional critical behaviour, whose results parallel and complement those of CFT. However, the new approach focused on the properties of a single such curve, conditioned to start at the boundary of the domain, in the background of all the others. This leads to a very specific and physically clear notion of conformal invariance. Moreover, it was shown by Loewner [4] in the 1920s that any such curve in the plane which does not cross itself can be described by a dynamical process called Loewner evolution, in which the curve is imagined to be grown in a continuous fashion. Instead of describing this process directly, Loewner considered the evolution of the analytic function which conformally maps the region outside the curve into a standard domain. This evolution, and therefore the curve itself, turns out to be completely determined by a real continuous function a_t. For random curves, a_t itself is random. (The notation a_t is used rather than $a(t)$ to conform to standard usage in the case when it is a stochastic variable.) Schramm [5] argued that, if the measure on the curve is to be conformally invariant in the precise sense referred to above, the only possibility is that a_t be one-dimensional Brownian motion, with only a single parameter left undetermined, namely the diffusion constant κ. This leads to stochastic-, or Schramm-, Loewner evolution (SLE). (In the original papers by Schramm et al. the term 'stochastic' was used. However, in the subsequent literature the '*S*' has often been taken to stand for Schramm in recognition of his contribution.) It should apply to any critical statistical mechanics model in which it is possible to identify these non-crossing paths on the lattice, as long as their continuum limits obey the underlying conformal invariance property. For only a few cases, including percolation, has it been proved that this property holds, but it is believed to be true for suitably defined curves in a whole class of systems known as $\mathrm{O}(n)$ models. Special cases, apart from percolation, include the Ising model, Potts models, the XY model, and self-avoiding walks. They each correspond to a particular choice of κ.

Starting from the assumption that SLE describes such a single curve in one of these systems, many properties, such as the values of many of the critical exponents, as well as the crossing formula mentioned above, have been rigorously derived in a brilliant series of papers by Lawler, Schramm, and Werner (LSW) [6]. Together with Smirnov's proof [7] of the conformal invariance property for the continuum limit of site percolation on the triangular lattice, they give a *rigorous* derivation [8] of the values of the critical exponents for two-dimensional percolation. This represents a paradigm shift in rigorous statistical mechanics, in that results are now being derived directly in the continuum for models for which the traditional lattice methods have, so far, failed.

However, from the point of view of theoretical physics, these advances are perhaps not so important for being rigorous, as for the new light they throw on the nature of the critical state, and on conformal field theory. In the CFT of the $O(n)$ model, the point where a random curve hits the boundary corresponds to the insertion of a local operator which has a particularly simple property: its correlation functions satisfy linear second-order differential equations [9]. These equations turn out to be directly related to the Fokker–Planck type equations one gets from the Brownian process which drives SLE. Thus, there is a close connection, at least at an operational level, between CFT and SLE. This has been made explicit in a series of papers by Bauer and Bernard [10] (see also [11]). Other fundamental concepts of CFT, such as the central charge c, have their equivalence in SLE. This is a rapidly advancing subject, and some of the more recent directions will be mentioned in the concluding section of this article.

1.2. Aims of this article

The original papers on SLE are mostly both long and difficult, using, moreover, concepts and methods foreign to most theoretical physicists. There are reviews, in particular those by Werner [12] and by Lawler [13] which cover much of the important material in the original papers. These are however written for mathematicians. A more recent review by Kager and Nienhuis [14] describes some of the mathematics in those papers in way more accessible to theoretical physicists, and should be essential reading for any reader who wants then to tackle the mathematical literature. A complete bibliography up to 2003 appears in [15].

However, the aims of the present article are more modest. First, it does not claim to be a thorough review, but rather a semi-pedagogical introduction. In fact some of the material, presenting some of the existing results from a slightly different, and hopefully clearer, point of view, has not appeared before in print. The article is directed at the theoretical physicist familiar with the basic concepts of quantum field theory and critical behaviour at the level of a standard graduate textbook, and with a theoretical physicist's knowledge of conformal mappings and stochastic processes. It is not the purpose to prove anything, but rather to describe the concepts and methods of SLE, to relate them to other ideas in theoretical physics, in particular CFT, and to illustrate them with a few simple computations, which, however, will be presented in a thoroughly non-rigorous manner. Thus, this review is most definitely not for mathematicians interested in learning about SLE, who will no doubt cringe at the lack of preciseness in some of the arguments and perhaps be puzzled by the particular choice of material. The notation used will be that of theoretical physics, for example $\langle\cdots\rangle$ for expectation value, and so will the terminology. The word 'martingale' has just made its only appearance. Perhaps the largest omission is any account of the central arguments of LSW [6] which relate SLE to various aspects of Brownian motion and thus allow for the direct computation of many critical exponents. These methods are in fact related to two-dimensional quantum gravity, whose role in this is already the subject of a recent long article by Duplantier [16].

2. Random curves and lattice models

2.1. The Ising and percolation models

In this section, we introduce the lattice models which can be interpreted in terms of random non-intersecting paths on the lattice whose continuum limit will be described by SLE.

The prototype is the Ising model. It is most easily realised on a honeycomb lattice (see Fig. 1). At each site r is an Ising 'spin' $s(r)$ which takes the values ± 1. The partition function is

$$Z_{\text{Ising}} = \text{Tr}\,\exp\left(\beta J \sum_{rr'} s(r)s(r')\right) \propto \text{Tr}\prod_{rr'}(1 + xs(r)s(r')), \tag{1}$$

where $x = \tanh \beta J$, and the sum and product are over all edges joining nearest neighbour pairs of sites. The trace operation is defined as $\text{Tr} = \prod_r (\frac{1}{2}\sum_{s(r)})$, so that $\text{Tr}\,s(r)^n = 1$ if n is even, and 0 if it is odd.

At high temperatures ($\beta J \ll 1$) the spins are disordered, and their correlations decay exponentially fast, while at low temperatures ($\beta J \gg 1$) there is long-range order: if the spins on the boundary are fixed say, to the value $+1$, then $\langle s(r)\rangle \neq 0$ even in the infinite volume limit. In between, there is a critical point. The conventional approach to the Ising model focuses on the behaviour of the correlation functions of the spins. In the scaling limit, they become local operators in a quantum field theory (QFT). Their correlations are power-law behaved at the critical point, which corresponds to a massless QFT, that is a conformal field theory (CFT). From this point of view (as well as exact lattice calculations) it is found that correlation functions like $\langle s(r_1)s(r_2)\rangle$ decay at large separations according to power laws $|r_1 - r_2|^{-2x}$: one of

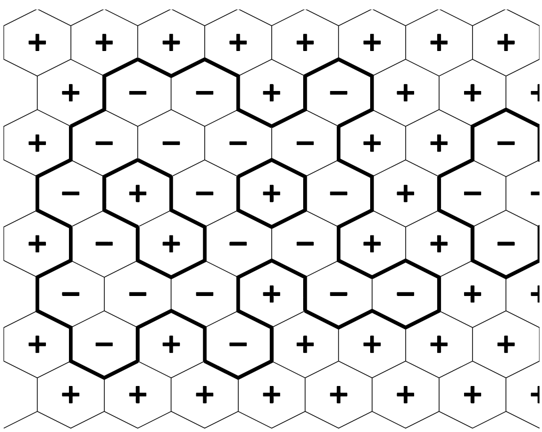

Fig. 1. Ising model on the honeycomb lattice, with loops corresponding to a term in the expansion of (1). Alternatively, these may be thought of as domain walls of an Ising model on the dual triangular lattice.

the aims of the theory is to obtain the values of the exponents x as well as to compute, for example, correlators depending on more than two points.

However, there is an alternative way of thinking about the partition function (1), as follows: imagine expanding out the product to obtain 2^N terms, where N is the total number of edges. Each term may be represented by a subset of edges, or graph $\mathscr{G}$, on the lattice, in which, if the term $xs(r)s(r')$ is chosen, the corresponding edge (rr') is included in $\mathscr{G}$, otherwise it is not. Each site r has either 0, 1, 2, or 3 edges in $\mathscr{G}$. The trace over $s(r)$ gives 1 if this number is even, and 0 if it is odd. Each surviving graph is then the union of non-intersecting closed loops (see Fig. 1). In addition, there can be open paths beginning and ending at a boundary. For the time being, we suppress these by imposing 'free' boundary conditions, summing over the spins on the boundary. The partition function is then

$$Z_{\text{Ising}} = \sum_{\mathscr{G}} x^{\text{length}}, \tag{2}$$

where the length is the total of all the loops in $\mathscr{G}$. When x is small, the mean length of a single loop is small. The critical point x_c is signalled by a divergence of this quantity. The low-temperature phase corresponds to $x > x_c$. While in this phase the Ising spins are ordered, and their connected correlation functions decay exponentially, the loop gas is in fact still critical, in that, for example, the probability that two points lie on the same loop has a power-law dependence on their separation. This is the *dense* phase.

The loops in $\mathscr{G}$ may be viewed in another way: as *domain walls* for another Ising model on the dual lattice, which is a triangular lattice whose sites R lie at the centres of the hexagons of the honeycomb lattice (see Fig. 1). If the corresponding interaction strength of this dual Ising model is $(\beta J)^*$, then the Boltzmann weight for creating a segment of domain wall is $\mathrm{e}^{-2(\beta J)*}$. This should be equated to $x = \tanh(\beta J)$ above. Thus, we see that the high-temperature regime of the dual model corresponds to low temperature in the original model, and vice versa. Infinite temperature in the dual model ($(\beta J)^* = 0$) means that the dual Ising spins are independent random variables. If we colour each dual site with $s(R) = +1$ black, and white if $s(R) = -1$, we have the problem of *site percolation* on the triangular lattice, critical because $p_c = \frac{1}{2}$ for that problem. Thus, the curves with $x = 1$ correspond to percolation cluster boundaries. (In fact in the scaling limit this is believed to be true throughout the dense phase $x > x_c$.)

So far we have discussed only closed loops. Consider the spin–spin correlation function

$$\langle s(r_1)s(r_2)\rangle = \frac{\mathrm{Tr}\, s(r_1)s(r_2)\prod_{rr'}(1 + xs(r)s(r'))}{\mathrm{Tr}\prod_{rr'}(1 + xs(r)s(r'))}, \tag{3}$$

where the sites r_1 and r_2 lie on the boundary. Expanding out as before, we see that the surviving graphs in the numerator each have a single edge coming into r_1 and r_2. There is therefore a single open path γ connecting these points on the boundary (which does not intersect itself nor any of the closed loops). In terms of the dual variables, such a single open curve may be realised by specifying the spins $s(R)$ on all the dual sites on the boundary to be +1 on the part of the boundary between r_1 and r_2 (going clockwise) and −1 on the remainder. There is then a single domain wall connecting r_1 to r_2. *SLE describes the continuum limit of such a curve* γ.

Note that we could also choose r_2 to lie in the interior. The continuum limit of such curves is then described by radial SLE (Section 3.6).

2.1.1. Exploration process

An important property of the ensemble of curves γ on the lattice is that, instead of generating a configuration of all the $s(R)$ and then identifying the curve, it may be constructed step-by-step as follows (see Fig. 2). Starting from r_1, at the next step it should turn R or L according to whether the spin in front of it is +1 or −1. For independent percolation, the probability of either event is 1/2, but for $x < 1$ it depends on the values of the spins on the boundary. Proceeding like this, the curve will grow, with all the dual sites on its immediate left taking the value +1, and those its right the value −1. The relative probabilities of the path turning R or L at a given step depend on the expectation value of the spin on the site R immediately in front of it, given the values of the spins already determined, that is, given the path up to that point. Thus, the relative probabilities that the path turns R or L are completely determined by the domain and the path up to that point. This implies the crucial.

Property 2.1 (Lattice version). *Let γ_1 be the part of the total path covered after a certain number of steps. Then the conditional probability distribution of the remaining part of the curve, given γ_1, is the same as the unconditional distribution of a whole curve, starting at the tip and ending at r_2 in the domain $\mathcal{D} \setminus \gamma_1$.*

In the Ising model, for example, if we already know part of the domain wall, the rest of it can be considered as a complete domain wall in a new region in which the left and right sides of the existing part form part of the boundary. This means the

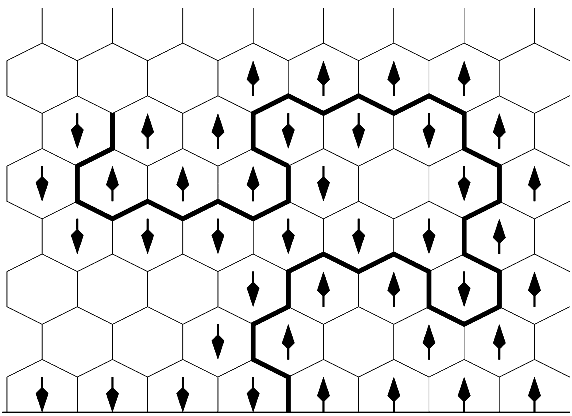

Fig. 2. The exploration process for the Ising model. At each step the walk turns L or R according to the value of the spin in front of it. The relative probabilities are determined by the expectation value of this spin given the fixed spins either side of the walk up to this time. The walk never crosses itself and never gets trapped.

path is a history-dependent random walk. It can be seen (Fig. 2) that when the growing tip τ approaches an earlier section of the path, it must always turn away from it: the tip never gets trapped. There is always at least one path on the lattice from the tip τ to the final point r_2.

2.2. $O(n)$ model

The loop gas picture of the Ising and percolation models may simply be generalised by counting each closed loop with a fugacity n

$$Z_{\mathrm{O}(n)} = \sum_{\mathscr{G}} x^{\text{length}} n^{\text{number of loops}}. \tag{4}$$

This is called the $\mathrm{O}(n)$ model, for the reason that it gives the partition function for n-component spins $\mathbf{s}(r) = (s_1(r), \ldots, s_n(r))$ with

$$Z_{\mathrm{O}(n)} = \mathrm{Tr} \prod_{rr'} (1 + x\mathbf{s}(r) \cdot \mathbf{s}(r')), \tag{5}$$

where $\mathrm{Tr}\, s_a(r) s_b(r) = \delta_{ab}$. Following the same procedure as before we obtain the same set of closed loops (and open paths) except that, on summing over the last spin in each closed loop, we get a factor n. The model is called $\mathrm{O}(n)$ because of its symmetry under rotations of the spins. The version (5) makes sense only when n is a positive integer (and note that the form of the partition function is different from that of the conventional $\mathrm{O}(n)$ model, where the second term is exponentiated). The form in (4) is valid for general values of n, and it gives a probability measure on the loop gas for real $n \geqslant 0$. However, the dual picture is useful only for $n = 1$ and $n = 2$ (see below). As for the case $n = 1$, there is a critical value $x_c(n)$ at which the mean loop length diverges. Beyond this, there is a dense phase.

Apart from $n = 1$, other important physical values of n are:

- $n = 2$. In this case we can view each loop as being oriented in either a clockwise or anti-clockwise sense, giving it an overall weight 2. Each loop configuration then corresponds to a configuration of integer valued *height* variables $h(R)$ on the dual lattice, with the convention that the nearest neighbour difference $h(R') - h(R)$ takes the values 0, $+1$ or -1 according to whether the edge crossed by RR' is unoccupied, occupied by an edge oriented at 90° to RR', or at $-90°$. (That is, the current running around each loop is the lattice curl of h.) The variables $h(R)$ may be pictured as the local height of a crystal surface. In the low-temperature phase (small x) the surface is smooth: fluctuations of the height differences decay exponentially with separation. In the high-temperature phase it is rough: they grow logarithmically. In between is a roughening transition. It is believed that relaxing the above restriction on the height difference $h(R') - h(R)$ does not change the universality class, as long as large values of this difference are suppressed, for example using the weighting $\exp[-\beta(h(R') - h(R))^2]$. This is the discrete Gaussian model. It is dual to a model of two-component spins with $\mathrm{O}(2)$ symmetry called the XY model.

- $n=0$. In this case, closed loops are completely suppressed, and we have a single non-self-intersecting path connecting r_1 and r_2, weighted by its length. Thus, all paths of the same length are counted with equal weight. This is the self-avoiding walk problem, which is supposed to describe the behaviour of long flexible polymer chains. As $x \to x_{c^-}$, the mean length diverges. The region $x > x_c$ is the dense phase, corresponding to a long polymer whose length is of the order of the area of the box, so that it has finite density.
- $n=-2$ corresponds to the loop-erased random walk. This is an ordinary random walk in which every loop, once it is formed, is erased. Taking $n=-2$ in the $\mathrm{O}(n)$ model of non-intersecting loops has this effect.

2.3. Potts model

Another important model which may described in terms of random curves in the Q-state Potts model. This is most easily considered on square lattice, at each site of which is a variable $s(r)$ which can take Q (initially a positive integer) different values. The partition function is

$$Z_{\mathrm{Potts}} = \mathrm{Tr}\,\exp\left(\beta J \sum_{rr'} \delta_{s(r),s(r')}\right) \propto \mathrm{Tr} \prod_{rr'} (1-p+p\delta_{s(r),s(r')}) \tag{6}$$

with $\mathrm{e}^{\beta J} = (1-p)^{-1}$. The product may be expanded in a similar way to the case of the Ising model. All possible graphs $\mathscr{G}$ will appear. Within each connected component of $\mathscr{G}$ the Potts spins must be equal, giving rise to a factor Q when the trace is performed. The result is

$$Z_{\mathrm{Potts}} = \sum_{\mathscr{G}} p^{|\mathscr{G}|}(1-p)^{|\overline{\mathscr{G}}|} Q^{\|\mathscr{G}\|}, \tag{7}$$

where $|\mathscr{G}|$ is the number of edges in $\mathscr{G}$, $|\overline{\mathscr{G}}|$ is the number in its complement, and $\|\mathscr{G}\|$ is the number of connected components of $\mathscr{G}$, which are called Fortuin–Kasteleyn (FK) clusters. This is the *random cluster* representation of the Potts model. When p is small, the mean cluster size is small. As $p \to p_c$, it diverges, and for $p > p_c$ there is an infinite cluster which contains a finite fraction of all the sites in the lattice. It should be noted that these FK clusters are *not* the same as the spin clusters within which the original Potts spins all take the same value.

The limit $Q \to 1$ gives another realisation of percolation—this time bond percolation on the square lattice. For $Q \to 0$ there is only one cluster. If at the same time $x \to 0$ suitably, all loops are suppressed and the only graphs $\mathscr{G}$ which contribute are *spanning trees*, which contain every site of the lattice. In the Potts partition function each possible spanning tree is counted with the same weight, corresponding to the problem of uniform spanning trees (UST). The ensemble of paths on USTs connecting two points r_1 and r_2 turns out to be be that of loop-erased random walks.

The random cluster model may be realised as a gas of dense loops in the way illustrated in Fig. 3. These loops lie on the medial lattice, which is also square but has

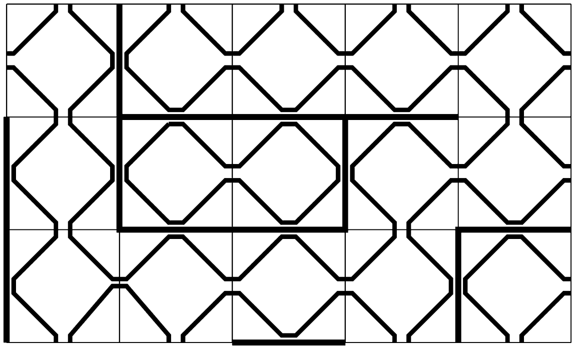

Fig. 3. Example of FK clusters (heavy lines) in the random cluster representation of the Potts model, and the corresponding set of dense loops (medium heavy) on the medial lattice. The loops never cross the edges connecting sites in the same cluster.

twice the number of sites. It may be shown that, at p_c, the weights for the clusters are equivalent to counting each loop with a fugacity $\sqrt{Q}$. Thus, the boundaries of the critical FK clusters in the Q-state Potts model are the same in the scaling limit (if it exists) as the closed loops of the dense phase of the $\mathrm{O}(n)$ model, with $n = \sqrt{Q}$.

To generate an open path in the random cluster model connecting sites r_1 and r_2 on the boundary we must choose 'wired' boundary conditions, in which $p = 1$ on all the edges parallel to the boundary, from r_1 to r_2, and free boundary conditions, with $p = 0$, along the remainder.

2.4. Coulomb gas methods

Many important results concerning the $\mathrm{O}(n)$ model can be derived in a non-rigorous fashion using so-called Coulomb gas methods. For the purposes of comparison with later results from SLE, we now summarise these methods and collect a few relevant formulae. A much more complete discussion may be found in the review by Nienhuis [3].

We assume that the boundary conditions on the $\mathrm{O}(n)$ spins are free, so that the partition function is a sum over closed loops only. First orient each loop at random. Rather than giving clockwise and anti-clockwise orientations the same weight $n/2$, give them complex weights $e^{\pm 6i\chi}$, where $n = e^{6i\chi} + e^{-6i\chi} = 2\cos 6\chi$. These may be taken into account, on the honeycomb lattice, by assigning a weight $e^{\pm i\chi}$ at each vertex where an oriented loop turns R (respectively, L). This transforms the non-local factors of n into local (albeit complex) weights depending only on the local configuration at each vertex.

Next transform to the height variables described above. By convention, the heights are taken to be integer multiples of π. The local weights at each vertex now depend only on the differences of the three adjacent heights. The crucial assumption of the Coulomb gas approach is that, under the RG, this model flows to one in which the lattice can be replaced by a continuum, and the heights go over into a gaussian free field, with partition function $Z = \int e^{-S[h]}[dh]$, where

$$S = (g/4\pi)\int (\nabla h)^2\,\mathrm{d}^2 r. \tag{8}$$

As it stands, this is a simple free field theory. The height fluctuations grow logarithmically: $\langle (h(r_1) - h(r_2))^2\rangle \sim (2/g)\ln|r_1 - r_2|$, and the correlators of exponentials of the height decay with power laws

$$\langle \mathrm{e}^{\mathrm{i}qh(r_1)}\mathrm{e}^{-\mathrm{i}qh(r_2)}\rangle \sim |r_1 - r_2|^{-2x_q}, \tag{9}$$

where $x_q = q^2/2g$. All the subtleties come from the combined effects of the phase factors and the boundaries or the topology. This is particularly easy to see if we consider the model on a cylinder of circumference ℓ and length $L \gg \ell$. In the simple gaussian model (8) the correlation function between two points a distance L apart along the cylinder decays as $\exp(-2\pi x_q L/\ell)$. However, if $\chi \neq 0$, loops which wrap around the cylinder are not counted correctly by the above prescription, because the total number of left turns minus right turns is then zero. We may arrange the correct factors by inserting $\mathrm{e}^{\pm 6\mathrm{i}\chi h/\pi}$ at either end of the cylinder. This has the effect of modifying the partition function: one finds $\ln Z \sim (\pi c/6)(L/\ell)$ with

$$c = 1 - 6\frac{(6\chi/\pi)^2}{g}. \tag{10}$$

This dependence of the partition function is one way of determining the so-called central charge of the corresponding CFT (Section 5). The charges at each end of the cylinder also modify the scaling dimension x_q to $(1/2g)((q - 6\mathrm{i}\chi/\pi)^2 - (6\mathrm{i}\chi/\pi)^2)$.

The value of g may be fixed [17] in terms of the original discreteness of the height variables as follows: adding a term $-\lambda\int\cos 2h\mathrm{d}^2 r$ to S in (8) ensures that, in the limit $\lambda \to \infty$, h will be an integer multiple of π. For this deformation not to affect the critical behaviour, it must be marginal in the RG sense, which means that it must have scaling dimension $x_2 = 2$. This condition then determines $g = 1 - 6\chi/\pi$.

2.4.1. Winding angle distribution

A simple property which can be inferred from the Coulomb gas formulation is the winding angle distribution. Consider a cylinder of circumference 2π and a path that winds around it. What the probability that it winds through an angle θ around the cylinder while it moves a distance $L \gg 1$ along the axis? This will correspond to a height difference $\Delta h = \pi(\theta/2\pi)$ between the ends of the cylinder, and therefore an additional free energy $(g/4\pi)(2\pi L)(\theta/2L)^2$. The probability density is therefore

$$P(\theta) \propto \exp(-g\theta^2/8L), \tag{11}$$

so that θ is normally distributed with variance $(4/g)L$. This result will be useful later (Section 3.6) for comparison with SLE.

2.4.2. N-leg exponent

As a final simple exponent prediction, consider the correlation function $\langle \Phi_N(r_1)\Phi_N(r_2)\rangle$ of the N-leg operator, which in the language of the $\mathrm{O}(n)$ model is $\Phi_N = s_{a_1}, \ldots, s_{a_N}$, where none of the indices are equal. It gives the probability that

N mutually non-intersecting curves connect the two points. Taking them a distance $L \gg \ell$ apart along the cylinder, we can choose to orient them all in the same sense, corresponding to a discontinuity in h of $N\pi$ in going around the cylinder. Thus, we can write $h = \pi N v/\ell + \tilde{h}$, where $0 \leqslant v < \ell$ is the coordinate around the cylinder, and $\tilde{h}(v+\ell) = \tilde{h}(v)$. This gives

$$\langle \Phi_N(r_1)\Phi_N(r_2)\rangle \sim \exp(-(g/4\pi)(N\pi/\ell)^2 L + (\pi L/6\ell) - (\pi c L/6\ell)). \tag{12}$$

The second term in the exponent comes from the integral over the fluctuations $\tilde{h}$, and the last from the partition function. They differ because in the numerator, once there are curves spanning the length of the cylinder, loops around it, which give the correction term in (10), are forbidden. Eq. (12) then gives

$$x_N = (gN^2/8) - (g-1)^2/2g. \tag{13}$$

3. SLE

3.1. The postulates of SLE

SLE gives a description of the continuum limit of the lattice curves connecting two points on the boundary of a domain $\mathcal{D}$ which were introduced in Section 2. The idea is to define a *measure* $\mu(\gamma;\mathcal{D}, r_1, r_2)$ on these continuous curves. (Note that the notion of a probability density of such objects does not make sense, but the more general concept of a measure does.)

There are two basic properties of this continuum limit which must either be assumed, or, better, proven to hold for a particular lattice model. The first is the continuum version of Property 3.1:

Property 3.1 (Continuum version). *Denote the curve by γ, and divide it into two disjoint parts: γ_1 from r_1 to τ, and γ_2 from τ to r_2. Then the conditional measure $\mu(\gamma_2|\gamma_1;\mathcal{D}, r_1, r_2)$ is the same as $\mu(\gamma_2;\mathcal{D}\setminus\gamma_1, \tau, r_2)$.*

This property we expect to be true for the scaling limit of all such curves in the O(n) model (at least for $n \geqslant 0$), even away from the critical point. However, the second property encodes the notion of conformal invariance, and it should be valid, if at all, only at $x = x_c$ and, separately, for $x > x_c$.

Property 3.2 (Conformal invariance). *Let Φ be a conformal mapping of the interior of the domain $\mathcal{D}$ onto the interior of $\mathcal{D}'$, so that the points (r_1, r_2) on the boundary of $\mathcal{D}$ are mapped to points (r_1', r_2') on the boundary of $\mathcal{D}'$. The measure μ on curves in $\mathcal{D}$ induces a measure $\Phi * \mu$ on the image curves in $\mathcal{D}'$. The conformal invariance property states that this is the same as the measure which would be obtained as the continuum limit of lattice curves from r_1' to r_2' in $\mathcal{D}'$. That is*

$$(\Phi * \mu)(\gamma;\mathcal{D}, r_1, r_2) = \mu(\Phi(\gamma);\mathcal{D}', r_1', r_2'). \tag{14}$$

3.2. Loewner's equation

We have seen that, on the lattice, the curves γ may be 'grown' through a discrete exploration process. The Loewner process is the continuum version of this. Because of Property 3.2 it suffices to describe this in a standard domain $\mathcal{D}$, which is taken to be the upper half plane $\mathbf{H}$, with the points r_1 and r_2 being the origin and infinity, respectively.

The first thing to notice is that, although on the honeycomb lattice the growing path does not intersect itself, in the continuum limit it might (although it still should not cross itself.) This means that there may be regions enclosed by the path which are not on the path but nevertheless are not reachable from infinity without crossing it. We call the union of the set of such points, together with the curve itself, up to time t, the *hull* K_t. (This is a slightly different usage of this term from that in the physics percolation literature.) It is the complement of the connected component of the half plane which includes ∞, itself denoted by $\mathbf{H}\backslash K_t$. See Fig. 4.

Another property which often holds in the half-plane is that of *reflection invariance*: the distribution of lattice paths starting from the origin and ending at ∞ is invariant under $x \to -x$. For the lattice paths in the $\mathrm{O}(n)$ model discussed in Section 2.2 this follows from the symmetry of the underlying weights, but for the boundaries of the FK clusters in the Potts model it is a consequence of duality. Not all simple curves in lattice models have this property. For example, if we consider the three-state Potts model in which the spins on the negative and positive real axes are fixed to different values, there is a simple lattice curve which forms the outer boundary of the spin cluster containing the positive real axis. This is not the same as the boundary of the spin cluster containing the negative real axis, and it is not in general symmetric under reflections.

Since $\mathbf{H}\backslash K_t$ is simply connected, by the Riemann mapping theorem it can be mapped into the standard domain $\mathbf{H}$ by an analytic function $g_t(z)$. Because this preserves the real axis outside K_t it is in fact real analytic. It is not unique, but can be made so by imposing the behaviour as $z \to \infty$

$$g_t(z) \sim z + \mathrm{O}(1/z). \tag{15}$$

It can be shown that, as the path grows, the coefficient of $1/z$ is monotonic increasing (essentially it is the electric dipole moment of K_t and its mirror image in the real

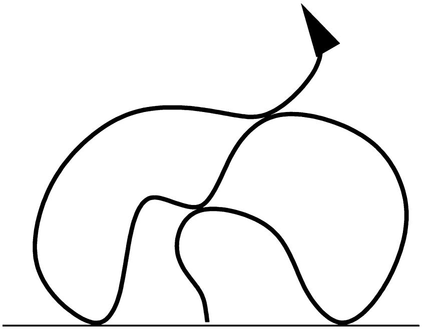

Fig. 4. Schematic view of a trace and its hull.

axis). Therefore, we may reparametrise time so that this coefficient is $2t$. (The factor 2 is conventional.) Note that the length of the curve is not be a useful parametrisation in the continuum limit, since the curve is a fractal.

The function $g_t(z)$ maps the whole boundary of K_t onto part of the real axis. In particular, it maps the growing tip τ_t to a real point a_t. Any point on the real axis outside K_t remains on the real axis. As the path grows, the point a_t moves on the real axis. For the path to describe a curve, it must always grow only at its tip, and this means that the function a_t must be continuous, but not necessarily differentiable.

A simple but instructive example is when γ is a straight line growing vertically upwards from a fixed point a. In this case

$$g_t(z) = a + ((z-a)^2 + 4t)^{1/2}. \tag{16}$$

This satisfies (15), and $\tau_t = 2\mathrm{i}\sqrt{t}$. More complicated deterministic examples can be found [18]. In particular, $a_t \propto t^{1/2}$ describes a straight line growing at a fixed angle to the real axis.

Loewner's idea [4] was to describe the path γ and the evolution of the tip τ_t in terms of the evolution of the conformal mapping $g_t(z)$. It turns out that the equation of motion for $g_t(z)$ is simple

$$\frac{\mathrm{d}g_t(z)}{\mathrm{d}t} = \frac{2}{g_t(z) - a_t}. \tag{17}$$

This is Loewner's equation. The idea of the proof is straightforward. Imagine evolving the path for a time t, and then for a further short time δt. The image of $K_{t+\delta t}$ under g_t is a short vertical line above the point a_t on the real axis. Thus, we can write, using (16)

$$g_{t+\delta t}(z) \approx a_t + ((g_t(z) - a_t)^2 + 4\delta t)^{1/2}. \tag{18}$$

Differentiating with respect to δt and then letting $\delta t \to 0$, we obtain (17).

Note that, even if a_t is not differentiable (as is the case for SLE), (17) gives for each point z_0 a solution $g_t(z_0)$ which is differentiable with respect to t, up to the time when $g_t(z_0) = a_t$. This is the time when z_0 is first included in K_t. However, it is sometimes (see Section 5) useful to normalise the Loewner mapping differently, defining $\hat{g}_t(z) = g_t(z) - a_t$, which always maps the growing tip τ_t to the origin. If a_t is not differentiable, neither is $\hat{g}_t$, and the Loewner equation should be written in differential form as $\mathrm{d}\hat{g}_t = (2\mathrm{d}t/\hat{g}_t) - \mathrm{d}a_t$.

Given a growing path, we can determine the hull K_t and hence, in principle, the function $g_t(z)$ and thereby $a_t = g_t(\tau_t)$. Conversely, given a_t we can integrate (17) to find $g_t(z)$ and hence in determine the curve (although proving that this inverse problem gives a curve is not easy).

3.3. Schramm–Loewner evolution

In the case that we are interested in, γ is a random curve, so that a_t is a random continuous function. What is the measure on a_t? This is answered by the following remarkable result, due to Schramm [5]:

Theorem 3.1. *If Properties* 3.1–3.2 *hold, together with reflection symmetry, then a_t is proportional to a standard Brownian motion.*

That is

$$a_t = \sqrt{\kappa} B_t, \tag{19}$$

so that $\langle a_t \rangle = 0$, $\langle (a_{t_1} - a_{t_2})^2 \rangle = \kappa |t_1 - t_2|$. The only undetermined parameter is κ, the diffusion constant. It will turn out that different values of κ correspond to different universality classes of critical behaviour.

The idea behind the proof is once again simple. As before, consider growing the curve for a time t_1, giving γ_1, and denote the remainder $\gamma \backslash \gamma_1 = \gamma_2$. Property 3.1 tells us that the conditional measure on γ_2 given γ_1 is the same as the measure on γ_2 in the domain $\mathbf{H} \setminus K_{t_1}$, which, by Property 3.2, induces the same measure on $g_{t_1}(\gamma_2)$ in the domain $\mathbf{H}$, shifted by a_{t_1}. In terms of the function a_t this means that the probability law of $a_t - a_{t_1}$, for $t > t_1$, is the same as the law of a_{t-t_1}. This implies that all the increments $\Delta_n \equiv a_{(n+1)\delta t} - a_{n\delta t}$ are independent identically distributed random variables, for all $\delta t > 0$. The only process that satisfies this is Brownian motion with a possible drift term: $a_t = \sqrt{\kappa} B_t + \alpha t$. Reflection symmetry then implies that $\alpha = 0$.

3.4. Simple properties of SLE

3.4.1. Phases of SLE

Many of the results discussed in this section have been proved by Rohde and Schramm [19]. First, we address the question of how the trace (the trajectory of τ_t) looks for different values of κ. For $\kappa = 0$, it is a vertical straight line. As κ increases, the trace should randomly turn to the L or R more frequently. However, it turns out that there are qualitative differences at critical values of κ. To see this, let us first study the process on the real axis. Let $x_t = g_t(x_0) - a_t$ be the distance between the image at time t of a point which starts at x_0 and the image a_t of the tip. It obeys the stochastic equation

$$\mathrm{d}x_t = \frac{2\mathrm{d}t}{x_t} - \sqrt{\kappa}\,\mathrm{d}B_t. \tag{20}$$

Physicists often write such an equation as $\dot{x} = (2/x) - \eta_t$ where η_t is 'white noise' of strength κ. Of course this does not make sense since x_t is not differentiable. Such equations are always to be interpreted in the 'Ito sense,' that is, as the limit as $\delta t \to 0$ of the forward difference equation $x_{t+\delta t} \approx x_t + (2\delta t / x_t) + \int_t^{t+\delta t} \eta_{t'}\,\mathrm{d}t'$.

Eq. (20) is known as the Bessel process. (If we set $R_t = (D-1)^{1/2} x_t / 2$ and $\kappa^2 = 4/(D-1)$ it describes the distance R_t from the origin of a Brownian particle in D dimensions.) The point x_t is repelled from the origin but it is also subject to a random force. Its ultimate fate can be inferred from the following crude argument: if we ignore the random force, $x_t^2 \sim 4t$, while, in the absence of the repulsive term, $\langle x_t^2 \rangle \sim \kappa t$. Thus, for $\kappa < 4$ the repulsive force wins and the particle escapes to infinity, while for $\kappa > 4$ the noise dominates and the particle collides with the origin in finite time (at which point the equation breaks down). A more careful analysis confirms this. What does this collision signify (see Fig. 5) in terms of the behaviour of the trace? In Fig. 6, we show a trace which is about to hit the real axis at the point x_0, thus engulfing a whole region. This is visible from infinity only through a very small opening, which means that, under g_t, it gets mapped to a very small region. In fact, as the tip τ_t approaches x_0, the size of the image of this region shrinks to zero. When the gap closes, the whole region enclosed by the trace, as well as τ_t and x_0, are mapped in to the single point a_t, which means, in particular, that $x_t \to 0$. The above argument shows that for $\kappa < 4$ this never happens: the trace never hits the real axis (with probability 1). For the same reason, it neither hits itself. Thus, for $\kappa < 4$ the trace γ is a *simple curve*.

The opposite is true for $\kappa > 4$: points on the real axis are continually colliding with the image a_t of the tip. This means that the trace is continually touching both itself and the real axis, at the same time engulfing whole regions. Moreover, since it is self-similar object, it does this on all scales, an infinite number of times within any finite neighbourhood! Eventually, the trace swallows the whole half plane: every point is ultimately mapped into a_t. For $\kappa < 4$ only the points on the trace itself suffer this fate. The case $\kappa = 4$ is more tricky: in fact the trace is then also a simple curve.

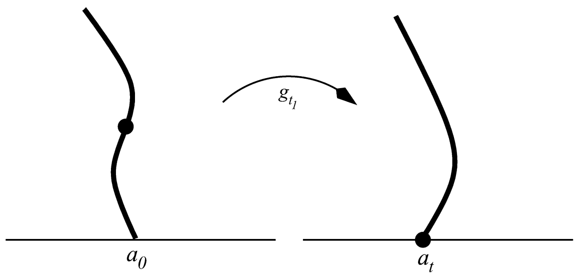

Fig. 5. A hull evolved from a_0 for time t_1, then to infinity. The measure on the image of the rest of the curve under g_{t_1} is the same, according to the postulates of SLE, as a hull evolved from a_{t_1} to ∞.

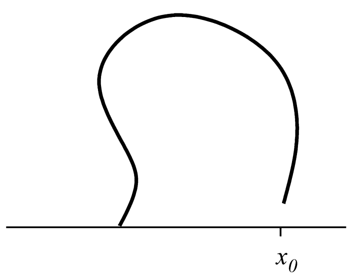

Fig. 6. The trace is about to hit the axis at x_0 and enclose a region. At the time this happens, the whole region including the point x_0 is mapped by g_t to the same point a_t.

When κ is just above 4, the images of points on the real axis under g_t collide with a_t only when there happen to be rare events when the random force is strong enough to overcome the repulsion. When this happens, whole segments of the real axis are swallowed at one time, corresponding to the picture described above. Conversely, for large κ, the repulsive force is negligible except for very small x_t. In that case, two different starting points move with synchronised Brownian motions until the one which started off closer to the origin is swallowed. Thus, the real line is eaten up in a continuous fashion rather than piecemeal. There are no finite regions swallowed by the trace which are not on the trace itself. This means that the trace is *space-filling*: γ intersects every neighbourhood of every point in the upper half plane. We shall argue later (Section 4.3.1) that the fractal dimension of the trace is $d_f = 1 + \kappa/8$ for $\kappa \leqslant 8$ and 2 for $\kappa \geqslant 8$. Thus, it becomes space-filling for all $\kappa \geqslant 8$.

3.4.2. SLE duality

For $\kappa > 4$ the curve continually touches itself and therefore its hull K_t contains earlier portions of the trace (see Fig. 4). However, the *frontier* of K_t (i.e., the boundary of $\mathbf{H}\backslash K_t$, minus any portions of the real axis), is by definition a simple curve. A beautiful result, first suggested by Duplantier [20], and proved by Beffara [21] for the case $\kappa = 6$, is that locally this curve is an $\mathrm{SLE}_{\tilde{\kappa}}$, with

$$\tilde{\kappa} = 16/\kappa. \tag{21}$$

For example, the exterior of a percolation cluster contains many 'fjords' which, on the lattice, are connected to the main ocean by a neck of water which is only a finite number of lattice spacings wide. These are sufficiently frequent and the fjords macroscopically large that they survive in the continuum limit. SLE_6 describes the boundaries of the clusters, including the coastline of all the fjords. However, the coastline as seen from the ocean is a simple curve, which is locally $\mathrm{SLE}_{8/3}$, the same as that conjectured for a self-avoiding walk. This suggests, for example, that locally the frontier of a percolation cluster and a self-avoiding walk are the same in the scaling limit. In Section 5, we show that SLE_κ and $\mathrm{SLE}_{\tilde{\kappa}}$ correspond to CFTs with the same value of the central charge c.

3.5. Special values of κ

3.5.1. Locality

(This subsection and the next are more technical and may be omitted at a first reading). We have defined SLE in terms of curves which connect the origin and infinity in the upper half plane. Property 3.2 then allows us to define it for any pair of boundary points in any simply connected domain, by a conformal mapping. It is interesting to study how the variation of the domain affects the SLE equation. Let A be a simply connected region connected to the real axis which is at some finite distance from the origin (see Fig. 7). Consider a trace γ_t, with hull K_t, which grows from the origin according to SLE in the domain $\mathbf{H}\backslash A$. According to Property 3.2, we can do this by first making a conformal mapping h_0 which removes A, and then a map $\tilde{g}_t$

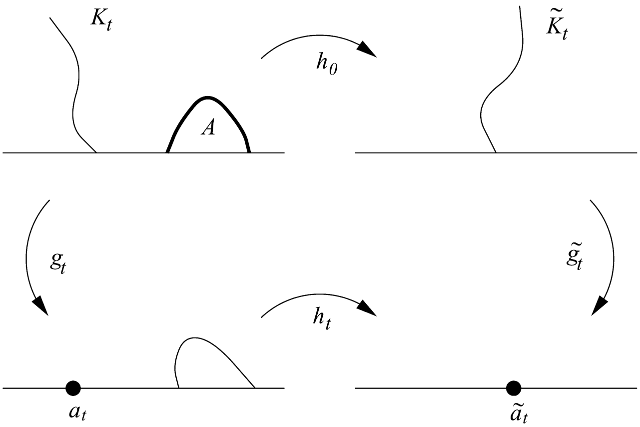

Fig. 7. An SLE hull in $\mathbf{H}\backslash A$ and two different ways of removing it: either by first removing A through h_0 and then using a Loewner map $\tilde{g}_t$ in the image of $\mathbf{H}\backslash A$; or by removing K_t first with g_t and then removing the image of A with h_t. Since all maps are normalised, this diagram commutes.

which removes the image $\tilde{K}_t = h_0(K_t)$. This would be described by SLE in $h_0(\mathbf{H}\backslash A)$, except that the Loewner 'time' would not in general be the same as t. However, another way to think about this is to first use a SLE map g_t in $\mathbf{H}$ to remove K_t, then another map, call it h_t, to remove $g_t(A)$. Since both these procedures end up removing $K_t \cup A$, and all the maps are assumed to be normalised at infinity in the standard way (15), they must be identical, that is $h_t \circ g_t = \tilde{g}_t \circ h_0$ (see Fig. 7). If g_t maps the growing tip τ_t to a_t, then after both mappings it goes to $\tilde{a}_t = h_t(a_t)$. We would like to understand the law of $\tilde{a}_t$.

Rather than working this out in full generality (see for example [12]), let us suppose that A is a short vertical segment $(x, x + i\epsilon)$ with $\epsilon \ll x$, and that $t = \mathrm{d}t$ is infinitesimal. Then, under $g_{\mathrm{d}t}$, $x \to x + 2\mathrm{d}t/x$ and $\epsilon \to \epsilon(1 - 2\mathrm{d}t/x^2)$. The map that removes this is (see (16))

$$h_{\mathrm{d}t}(z) = ((z - x - 2\mathrm{d}t/x)^2 + \epsilon^2(1 - 2\mathrm{d}t/x^2)^2)^{1/2} + x + 2\,\mathrm{d}t/x. \tag{22}$$

To find $\tilde{a}_{\mathrm{d}t}$, we need to set $z = a_{\mathrm{d}t} = \sqrt{\kappa}\mathrm{d}B_t$ in this expression. Carefully expanding this to first order in $\mathrm{d}t$, remembering that $(\mathrm{d}B_t)^2 = \mathrm{d}t$, and also taking the first non-zero contribution in ϵ/x, gives after a few lines of algebra

$$\tilde{a}_{\mathrm{d}t} = (1 - \epsilon^2/x^2)\sqrt{\kappa}\mathrm{d}B_t + \tfrac{1}{2}(\kappa - 6)(\epsilon^2/x^3)\,\mathrm{d}t. \tag{23}$$

The factor in front of the stochastic term may be removed by rescaling $\mathrm{d}t$: this restores the correct Loewner time. But there is also a drift term, corresponding to the effect of A. For $\kappa < 6$ we see that the SLE is initially repelled from A. From the point of view of the exploration process for the Ising model discussed in Section 2.1.1, this makes sense: if the spins along the positive real axis and on A are fixed to be up, then the spin just above the origin is more likely to be up than down, and so γ is more likely to turn to the left.

For $\kappa = 6$, however, this is no longer the case: the presence of A does not affect the initial behaviour of the curve. This is a particular case of the property of *locality* when $\kappa = 6$, which states that, for any A as defined above, the law of K_t in $\mathbf{H}\backslash A$ is, up to a time reparametrisation, the same as the law of K_t in $\mathbf{H}$, as long as $K_t \cap A = \emptyset$. That is, up to the time that the curve hits A, it does not know its there. Such a property would be expected for the cluster boundaries of uncorrelated Ising spins on the lattice, i.e., percolation. This is then consistent with the identification of percolation cluster boundaries with SLE_6.

3.5.2. Restriction

It is also interesting to work out how the local scale transforms in going from a_t to $\tilde{a}_t$. A measure of this is $h_t'(a_t)$. A similar calculation starting from (22) gives, in the same limit as above

$$\begin{aligned} \mathrm{d}(h'(a_t)) &= h'_{\mathrm{d}t}(a_{\mathrm{d}t}) - h_0'(0) \\ &= (\epsilon^2/x^3)\sqrt{\kappa}\mathrm{d}B_t + \tfrac{1}{2}\big((\epsilon^4/x^6) + (\kappa - \tfrac{8}{3})(3\epsilon^2/x^4)\big). \end{aligned} \tag{24}$$

Now something special happens when $\kappa = 8/3$. The drift term in $\mathrm{d}(h'(a_t))$ does not then vanish, but if we take the appropriate power $\mathrm{d}(h_t'(a_t)^{5/8})$ it does. This implies that the *mean* of $h_t'(a_t)^{5/8}$ is conserved. Now at $t = 0$ it takes the value $\Phi_A'(0)^{5/8}$, where $\Phi_A = h_0$ is the map that removes A. If K_t hits A at time T it can be seen from (22) that $\lim_{t\to T} h_t'(a_t)^{5/8} = 0$. On the other hand, if it never hits A then $\lim_{t\to\infty} h_t'(a_t)^{5/8} = 1$. Therefore, $\Phi_A'(0)^{5/8}$ gives the *probability that the curve γ does not intersect A*.

This is a remarkable result in that it depends only on the value of Φ_A' at the starting point of the SLE (assuming of course that Φ_A is correctly normalised at infinity). However, it has the following even more interesting consequence. Let $\hat{\Phi}_A(z) = \Phi_A(z) - \Phi(0)$. Consider the ensemble of all $\mathrm{SLE}_{8/3}$ in $\mathbf{H}$, and the sub-ensemble consisting of all those curves γ which do not hit A. Then the measure on the image $\hat{\Phi}_A(\gamma)$ in $\mathbf{H}$ is again given by $\mathrm{SLE}_{8/3}$. The way to show this is to realise that the measure on γ is characterised by the probability $P(\gamma \cap A' = \emptyset)$ that γ does not hit A' for all possible A'. The probability that $\hat{\Phi}_A(\gamma)$ does not hit A', given that γ does not hit A, is the ratio of the probabilities $P(\gamma \cap \hat{\Phi}_A^{-1}(A') = \emptyset)$ and $P(\gamma \cap A = \emptyset)$. By the above result, the first factor is the derivative at the origin of the map $\hat{\Phi}_{A'} \circ \hat{\Phi}_A$ which removes A then A', while the second is the derivative of the map which removes A. Thus

$$\begin{aligned} P(\hat{\Phi}_A(\gamma) \cap A' = \emptyset | \gamma \cap A = \emptyset) &= \left(\frac{(\hat{\Phi}_{A'} \circ \hat{\Phi}_A)'(0)}{\hat{\Phi}_A'(0)}\right)^{5/8} = \hat{\Phi}_{A'}'(0)^{5/8} \\ &= P(\gamma \cap A' = \emptyset). \end{aligned} \tag{25}$$

Since this is true for all A', it follows that the law of $\hat{\Phi}_A(\gamma)$ given that γ does not intersect A is the same as that of γ. This is called the *restriction property*. Note that while, according to Property 3.2, the law of an SLE in any simply connected subset of $\mathbf{H}$ is determined by the conformal mapping of this subset to $\mathbf{H}$, the restriction property is stronger than this, and it holds only when $\kappa = 8/3$.

We expect such a property to hold for the continuum limit of self-avoiding walks, assuming it exists. On the lattice, every walk of the same length is counted with the same weight. That is, the measure is uniform. If we consider the sub-ensemble of such walks which avoid a region A, the measure on the remainder should still be uniform. This will be true if the restriction property holds. This supports the identification of self-avoiding walks with $\mathrm{SLE}_{8/3}$.

3.6. Radial SLE and the winding angle

So far we have discussed a version of SLE that gives a conformally invariant measure on curves which connect two distinct boundary points of a simply connected domain $\mathscr{D}$. For this reason it is called *chordal* SLE. There are variants which describe other situations. For example, one could consider curves γ which connect a given point r_1 on the boundary to an interior point r_2. The Riemann mapping theorem allows us to map conformally onto another simple connected domain, with r_2 being mapped to any preassigned interior point. It is simplest to choose for the standard domain the unit disc $\mathbf{U}$, with r_2 being mapped to the origin. So we are considering curves γ which connect a given point $\mathrm{e}^{\mathrm{i}\theta_0}$ on the boundary with the origin. As before, we may consider growing the curve dynamically. Let K_t be the hull of that portion which exists up to time t. Then there exists a conformal mapping g_t which takes $\mathbf{U}\backslash K_t$ to $\mathbf{U}$, such that $g_t(0) = 0$. There is one more free parameter, which corresponds to a global rotation: we use this to impose the condition that $g_t'(0)$ is real and positive. One can then show that, as the curve grows, this quantity is monotonically increasing, and we can use this to reparametrise time so that $g_t'(0) = \mathrm{e}^t$. This normalised mapping then takes the growing tip τ_t to a point $\mathrm{e}^{\mathrm{i}\theta_t}$ on the boundary.

Loewner's theorem tells us that $\dot{g}_t(z)/g_t(z)$, when expressed as a function of $g_t(z)$, should be holomorphic in $\overline{\mathbf{U}}$ apart from a simple pole at $\mathrm{e}^{\mathrm{i}\theta_t}$. Since g_t preserves the unit circle outside K_t, $\dot{g}_t(z)/g_t(z)$ should be pure imaginary when $|g_t(z)| = 1$, and in order that $g_t'(0) = \mathrm{e}^t$, it should approach 1 as $g_t(z) \to 0$. The only possibility is

$$\frac{\mathrm{d}g_t(z)}{\mathrm{d}t} = -g_t(z)\frac{g_t(z) + \mathrm{e}^{\mathrm{i}\theta_t}}{g_t(z) - \mathrm{e}^{\mathrm{i}\theta_t}}. \tag{26}$$

This is the radial Loewner equation. In fact this is the version considered by Löewner [4].

It can now be argued, as before, that given Properties 3.1 and 3.2 (suitably reworded to cover the case when r_2 is an interior point) together with reflection, θ_t must be proportional to a standard Brownian motion. This defines radial SLE. It is not immediately obvious how the radial and chordal versions are related. However, it can be shown that, if the trace of radial SLE hits the boundary of the unit disc at $\mathrm{e}^{\mathrm{i}\theta_{t_1}}$ at time t_1, then the law of K_t in radial SLE, for $t < t_1$, is the same chordal SLE conditioned to begin at $\mathrm{e}^{\mathrm{i}\theta(0)}$ and end at $\mathrm{e}^{\mathrm{i}\theta_{t_1}}$, up to a reparametrisation of time. This assures us that, in using the chordal and radial versions with the same κ, we are describing the same physical problem.

However, one feature that the trace of radial SLE possesses which chordal SLE does not is the property that it can wind around the origin. The winding angle at time

t is simply $\theta_t - \theta_0$. Therefore, it is normally distributed with variance κt. At this point we can make a connection to the Coulomb gas analysis of the $\mathrm{O}(n)$ model in Section 2.4.1, where it was shown that the variance in the winding angle on a cylinder of length L is asymptotically $(4/g)L$. A semi-infinite cylinder, parametrised by w, is conformally equivalent to the unit disc by the mapping $z = \mathrm{e}^{-w}$. Asymptotically, $\mathrm{Re}\, w \to \mathrm{Re}\, w - t$ under Loewner evolution. Thus, we can identify $L \sim t$ and hence

$$\kappa = 4/g. \tag{27}$$

3.6.1. Identification with lattice models

This result allows use to make a tentative identification with the various realisations of the $\mathrm{O}(n)$ model described in Section 2.2. We have, using (27), $n = -2\cos(4\pi/\kappa)$ with $2 \leqslant \kappa \leqslant 4$ describing the critical point at x_c, and $4 < \kappa \leqslant 8$ corresponding to the dense phase. Some important special cases are therefore:

- $\kappa = -2$: loop-erased random walks (proven in [24]);
- $\kappa = 8/3$: self-avoiding walks, as already suggested by the restriction property, Section 3.5.2; unproven, but see [22] for many consequences;
- $\kappa = 3$: cluster boundaries in the Ising model, however, as yet unproven;
- $\kappa = 4$: BCSOS model of roughening transition (equivalent to the 4-state Potts model and the double dimer model), as yet unproven; also certain level lines of a gaussian random field and the 'harmonic explorer' (proven in [23]); also believed to be dual to the Kosterlitz–Thouless transition in the XY model;
- $\kappa = 6$: cluster boundaries in percolation (proven in [7]);
- $\kappa = 8$: dense phase of self-avoiding walks; boundaries of uniform spanning trees (proven in [24]).

It should be noted that no lattice candidates for $\kappa > 8$, or for the dual values $\kappa < 2$, have been proposed. This possibly has to do with the fact that, for $\kappa > 8$, the SLE trace is not reversible: the law on curves from r_1 to r_2 is not the same as the law obtained by interchanging the points. Evidently, curves in equilibrium lattice models should satisfy reversibility.

4. Calculating with SLE

SLE shows that the measure on the continuum limit of single curves in various lattice models is given in terms of one-dimensional Brownian motion. However, it is not at all clear how thereby to deduce interesting physical consequences. We first describe two relatively simple computations in two-dimensional percolation which can be done using SLE.

4.1. Schramm's formula

Given a curve γ connecting two points r_1 and r_2 on the boundary of a domain $\mathcal{D}$, what is the probability that it passes to the left of a given interior point? This is not a question which is natural in conventional approaches to critical behaviour, but which is very simply answered within SLE [25].

As usual, we can consider $\mathcal{D}$ to be the upper half plane, and take $r_1 = a_0$ and r_2 to be at infinity. The curve is then described by chordal SLE starting at a_0. Label the interior point by the complex number ζ.

Denote the probability that γ passes to the left of ζ by $P(\zeta, \bar{\zeta}; a_0)$ (we include the dependence on $\bar{\zeta}$ to emphasise the fact that this is a not a holomorphic function). Consider evolving the SLE for an infinitesimal time dt. The function $g_{\mathrm{d}t}$ will map the remainder of γ into its image γ', which, however, by Properties 3.1 and 3.2, will have the same measure as SLE started from $a_{\mathrm{d}t} = a_0 + \sqrt{\kappa}\mathrm{d}B_t$. At the same time, $\zeta \to g_{\mathrm{d}t}(\zeta) = \zeta + 2\mathrm{d}t/(\zeta - a_0)$. Moreover, γ' lies to the left of ζ' iff γ lies to the left of ζ. Therefore

$$P(\zeta, \bar{\zeta}; a_0) = \langle P\big(\zeta + 2\mathrm{d}t/(\zeta - a_0), \bar{\zeta} + 2\mathrm{d}t/(\bar{\zeta} - a_0), a_0 + \sqrt{\kappa}\mathrm{d}B_t\big)\rangle, \tag{28}$$

where the average $\langle \ldots \rangle$ is over all realisations of Brownian motion dB_t up to time dt. Taylor expanding, using $\langle \mathrm{d}B_t \rangle = 0$ and $\langle (\mathrm{d}B_t)^2 \rangle = \mathrm{d}t$, and equating the coefficient of dt to zero gives

$$\left(\frac{2}{\zeta - a_0}\frac{\partial}{\partial\zeta} + \frac{2}{\bar{\zeta} - a_0}\frac{\partial}{\partial\bar{\zeta}} + \frac{\kappa}{2}\frac{\partial^2}{\partial a_0^2}\right) P(\zeta, \bar{\zeta}; a_0) = 0. \tag{29}$$

Thus, P satisfies a linear second-order partial differential equation, typical of conditional probabilities in stochastic differential equations.

By scale invariance P in fact depends only on the angle θ subtended between $\zeta - a_0$ and the real axis. Thus, (29) reduces to an ordinary second-order linear differential equation, which is in fact hypergeometric. The boundary conditions are that $P = 0$ and 1 when $\theta = \pi$ and 0, respectively, which gives (specialising to $\kappa = 6$)

$$P = \frac{1}{2} + \frac{\Gamma(\frac{2}{3})}{\sqrt{\pi}\Gamma(\frac{1}{6})}(\cot\theta)\,{}_2F_1\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{2}; -\cot^2\theta\right). \tag{30}$$

Note that this may also be written in terms of a single quadrature since one solution of (29) is $P = \mathrm{const}$.

4.2. Crossing probability

Given a critical percolation problem inside a simply connected domain $\mathcal{D}$, what is the probability that a cluster connects two disjoint segments AB and CD of the boundary? This problem was conjectured to be conformally invariant and (probably) first studied numerically in [26]. A formula based on CFT as well as a certain amount of guesswork was conjectured in [1]. It was proved, for the continuum limit of site percolation on the triangular lattice, by Smirnov [7].

Within SLE, it takes a certain amount of ingenuity [5] to relate this problem to a question about a single curve. As usual, let $\mathscr{D}$ be the upper half plane. It is always possible to make a fractional linear conformal mapping which takes AB into $(-\infty, x_1)$ and CD into $(0, x_2)$, where $x_1 < 0$ and $x_2 > 0$. Now go back to the lattice picture and consider critical site percolation on the triangular lattice in the upper half plane, so that each site is independently coloured black or white with equal probabilities 1/2. Choose all the boundary sites on the positive real axis to be white, all those on the negative real axis to be black (see Fig. 8). There is a cluster boundary starting at the origin, which, in the continuum limit, will be described by SLE_6. Since $\kappa > 4$, it repeatedly hits the real axis, both to the L and R of the origin. Eventually every point on the real axis is swallowed. Either x_1 is swallowed before x_2, or vice versa.

Note that every site on the L of the curve is black, and every site on its R is white. Suppose that x_1 is swallowed before x_2. Then, at the moment it is swallowed, there exists a continuous path on the white sites, just to the R of the curve, which connects $(0, x_2)$ to the row just above $(-\infty, x_1)$. On the other hand, if x_2 is swallowed before x_1, there exists a continuous path on the black sites, just to the L of the curve, connecting $0-$ to a point on the real axis to the R of x_2. This path forms a barrier (as in the game of Hex) to the possibility of a white crossing from $(0, x_2)$ to $(-\infty, x_1)$. Hence there is such a crossing if and only if x_1 is swallowed before x_2 by the SLE curve.

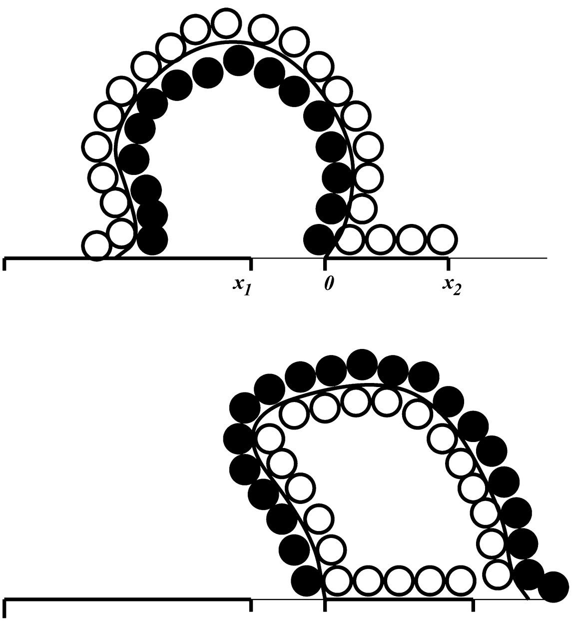

Fig. 8. Is there a crossing on the white discs from $(0, x_2)$ to $(-\infty, x_1)$? This happens if and only if x_1 gets swallowed by the SLE before x_2.

Recall that in Section 3.4.1 we related the swallowing of a point x_0 on the real axis to the vanishing of $x_t = g_t(x_t) - a_t$, which undergoes a Bessel process on the real line. Therefore

$$\Pr(\text{crossing from}(0,x_2)\text{ to }(-\infty,x_1)) = \Pr(x_{1t}\text{vanishes before }x_{2t}). \tag{31}$$

Denote this by $P(x_1,x_2)$. By generalising the SLE to start at a_0 rather than 0, we can write a differential equation for this in similar manner to (29)

$$\left(\frac{2}{x_1-a_0}\frac{\partial}{\partial x_1}+\frac{2}{x_2-a_0}\frac{\partial}{\partial x_2}+\frac{\kappa}{2}\frac{\partial^2}{\partial a_0^2}\right)P(x_1,x_2;a_0)=0. \tag{32}$$

Translational invariance implies that we can replace ∂_{a_0} by $-(\partial_{x_1}+\partial_{x_2})$. Finally, P can in fact depend only on the ratio $\eta = (x_2-a_0)/(a_0-x_1)$, which again reduces the equation to hypergeometric form. The solution is (specialising to $\kappa = 6$ for percolation)

$$P = \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{4}{3})\Gamma(\frac{1}{3})}\eta^{1/3}{}_2F_1\left(\frac{1}{3},\frac{2}{3},\frac{4}{3};\eta\right). \tag{33}$$

It should be mentioned that this is but one of a number of percolation crossing formulae. Another, conjectured by Watts [27], for the probability that there is cluster which simultaneously connects AB to CD and BC to DA, has since been proved by Dubédat [28]. However, other formulae, for example for the mean number of distinct clusters connecting AB and CD [29], and for the probability that exactly N distinct clusters cross an annulus [30], are as yet unproven using SLE methods.

4.3. Critical exponents from SLE

Many of the critical exponents which have previously been conjectured by Coulomb gas or CFT methods may be derived rigorously using SLE, once the underlying postulates are assumed or proved. However SLE describes the measure on just a single curve, and in the papers of LSW a great deal of ingenuity has gone into showing how to relate this to all the other exponents. There is not space in this article to do these justice. Instead we describe three examples which give the flavour of the arguments, which initially may appear quite unconventional compared with the more traditional approaches.

4.3.1. The fractal dimension of SLE

The fractal dimension of any geometrical object embedded in the plane can be defined roughly as follows: let $N(\epsilon)$ be the minimum number of small discs of radius ϵ required to cover the object. Then if $N(\epsilon) \sim \epsilon^{-d_f}$ as $\epsilon \to 0$, d_f is the fractal dimension.

One way of computing d_f for a random curve γ in the plane is to ask for the probability $P(x,y,\epsilon)$ that a given point $\zeta = x + \mathrm{i}y$ lies within a distance ϵ of γ. A simple scaling argument shows that if P behaves like $\epsilon^{\delta} f(x,y)$ as $\epsilon \to 0$, then $\delta = 2 - d_f$. We can derive a differential equation for P along the lines of the previous calculation. The only difference is that under the conformal mapping $g_{\mathrm{d}t}, \epsilon \to |g'_{\mathrm{d}t}(\zeta)|\epsilon \sim$

$(1 - 2\mathrm{d}t\,\mathrm{Re}(1/\zeta^2))\epsilon$. The differential equation (written for convenience in cartesian coordinates) is

$$\left(\frac{2x}{x^2+y^2}\frac{\partial}{\partial x} - \frac{2y}{x^2+y^2}\frac{\partial}{\partial y} + \frac{\kappa}{2}\frac{\partial^2}{\partial x^2} - \frac{2(x^2-y^2)}{(x^2+y^2)^2}\epsilon\frac{\partial}{\partial \epsilon}\right)P = 0. \tag{34}$$

Now P is dimensionless and therefore should have the form $(\epsilon/r)^{2-d_f}$ times a function of the polar angle θ. In fact, the simple ansatz $P = \epsilon^{2-d_f}y^\alpha(x^2+y^2)^\beta$, with $\alpha + 2\beta = d_f - 2$ satisfies the equation. [The reason this works is connected with the simple form for the correlator $\langle \Phi_2\phi_{2,1}\phi_{2,1}\rangle$ discussed in Section 5.4.1.] This gives $\alpha = (\kappa-8)^2/8\kappa$, $\beta = (\kappa-8)/2\kappa$, and

$$d_f = 1 + \kappa/8. \tag{35}$$

This is correct for $\kappa \leqslant 8$: otherwise there is another solution with $\alpha = \beta = 0$ and $d_f = 2$. A more careful statement and proof of this result can be found in [31].

We see that the fractal dimension increases steadily from the value 1 when $\kappa = 0$ (corresponding to a straight line) to a maximum value of 2 when $\kappa = 8$. Beyond this value γ becomes space-filling: every point in the upper half plane lies on the curve.

4.3.2. Crossing exponent

Consider a critical percolation problem in the upper half plane. What is the asymptotic behaviour as $r \to \infty$ of the probability that the interval $(0,1)$ on the real axis is connected to the interval (r,∞)? We expect this to decay as some power of r. The value of this exponent may be found by taking the appropriate limit of the crossing formula (33), but instead we shall compute it directly. In order for there to be a crossing cluster, there must be two cluster boundaries which also cross between the two intervals, and which bound this cluster above and below. Denote the upper boundary by γ. Then we need to know the probability $P(r)$ of there being another spanning curve lying between γ and $(1,r)$, averaged over all realisations of γ. Because of the locality property, the measure on γ is independent of the existence of the lower boundary, and is given by SLE_6 conditioned not to hit the real axis along $(1,r)$. Note that because $\kappa > 4$ it will eventually hit the real axis at some point to the right of r. For this reason we can do the computation for general $\kappa > 4$, although it gives the actual crossing exponent only if $\kappa = 6$.

Consider the behaviour of $P(r)$ under the conformal mapping $\hat{g}_{\mathrm{d}t}(z) \sim z + (2\mathrm{d}t/z) - \sqrt{\kappa}\mathrm{d}B_t$ (which maps the growing tip τ_t into 0). The crossing probability should be conformally invariant and depend only on the ratio of the lengths of the two intervals, hence, by an argument which by now should be familiar

$$P(r) = \langle P(\hat{g}_{\mathrm{d}t}(r)/\hat{g}_{\mathrm{d}t}(1))\rangle. \tag{36}$$

Expanding this out, remembering as usual that $(\mathrm{d}B_t)^2 = \mathrm{d}t$, and setting to zero the $\mathrm{O}(\mathrm{d}t)$ term, we find for $r \gg 1$

$$(\kappa-2)rP'(r) + \tfrac{1}{2}\kappa r^2 P''(r) = 0 \tag{37}$$

with the solution $P(r) \propto r^{-(\kappa-4)/\kappa}$ for $\kappa > 4$. Setting $\kappa = 6$ then gives the result 1/3.

4.3.3. The one-arm exponent

Consider critical lattice percolation inside some finite region (for example a disc of radius R). What is the probability that a given site (e.g., the origin) is connected to a finite segment S of the boundary? This should decay like $R^{-\lambda}$, where λ is sometimes called the one-arm exponent. If we try to formulate this in the continuum, we immediately run up against the problem that all clusters are fractal with dimension <2, and so the probability of any given point being in any given cluster is zero. Instead, one may ask about the probability $P(r)$ that the cluster connected to S gets within a distance r of the origin. This should behave like $(r/R)^{\lambda}$. We can now set $R=1$ and treat the problem using radial SLE_6.

Consider now a radial SLE_κ which starts at $\mathrm{e}^{\mathrm{i}\theta_0}$. If $\kappa>4$ it will continually hit the boundary. Let $P(\theta-\theta_0,t)$ be the probability that the segment (θ_0,θ) of the boundary has not been swallowed by time t. Then, by considering the evolution as usual under $g_{\mathrm{d}t}$

$$P(\theta,\theta_0,t)=\langle P(\theta+\mathrm{d}\theta,\theta_0+\mathrm{d}\theta_0,t-\mathrm{d}t)\rangle, \tag{38}$$

where $\mathrm{d}\theta=\cot((\theta-\theta_0)/2)\,\mathrm{d}t$ and $\mathrm{d}\theta_0=\sqrt{\kappa}\mathrm{d}B_t$. Setting $\theta_0=0$ and equating to zero the $\mathrm{O}(\mathrm{d}t)$ term, we find the time-dependent differential equation

$$\partial_t P=\cot(\theta/2)\partial_\theta P+\tfrac{1}{2}\kappa\partial_\theta^2 P. \tag{39}$$

This has the form of a backwards Fokker–Plank equation.

Now, since $g_t'(0)=\mathrm{e}^t$, it is reasonable that, after time t, the SLE gets within a distance $\mathrm{O}(\mathrm{e}^{-t})$ of the origin. Therefore, we can interpret P as roughly the probability that the cluster connected to $(0,\theta)$ gets within a distance $r\sim\mathrm{e}^{-t}$ of the origin. A more careful argument [32] confirms this. The boundary conditions are $P(0,t)=0$ as $\theta\to 0$, and (with more difficulty) $\partial_\theta P(\theta,t)=0$ at $\theta=2\pi$. The solution may then be found by inspection to be

$$P\propto \mathrm{e}^{-\lambda t}(\sin(\theta/4))^{1-4/\kappa}, \tag{40}$$

where $\lambda=(\kappa^2-16)/32\kappa$. For percolation this gives 5/48, in agreement with Coulomb gas arguments [3].

The appearance of differential operators such as that in (39) will become clear from the CFT perspective in Section 5.4.1. If instead of choosing Neuman boundary conditions at $\theta=2\pi$ we impose $P=0$, the same equation gives the bulk two-leg exponent x_2, which is also related to the fractal dimension by $d_f=2-x_2$.

5. Relation to conformal field theory

5.1. Basics of CFT

The reader who already knows a little about CFT will have recognised the differential equations in Section 4 as being very similar in form to the BPZ equations [33] satisfied by the correlation functions of a $\phi_{2,1}$ operator, corresponding to a highest weight representation of the Virasoro algebra with a level 2 null state.

For those readers for whom the above paragraph makes no sense, and in any case to make the argument self-contained, we first introduce the concepts of (boundary) conformal field theory (BCFT). We stress that these are heuristic in nature—they serve only as a guide to formulating the basic principles of CFT which can then be developed into a mathematically consistent theory. For a longer introduction to BCFT see [34] and, for a complete account of CFT [35].

We have at the back of our minds a euclidean field theory defined as a path integral over some set of fundamental fields $\{\psi(r)\}$. The partition function is $Z = \int e^{-S(\{\psi\})}[d\psi]$ where the action $S(\{\psi\}) = \int_{\mathscr{D}} \mathscr{L}(\{\psi\})\, d^2r$ is an integral over a local lagrangian density. These fields may be thought of as smeared-out continuum versions of the lattice degrees of freedom. As in any field theory, this continuum limit involves renormalisation. There are so-called local scaling operators $\phi_j^{(0)}(r)$ which are particular functionals of the fundamental degrees of freedom, which have the property that we can define renormalised scaling operators $\phi_j(r) = a^{-x_j}\phi_j^{(0)}(r)$ whose correlators are finite in the continuum limit $a \to 0$, that is

$$\lim_{a\to 0} a^{-\sum_j x_j} \langle \phi_1^{(0)}(r_1) \ldots \phi_N^{(0)}(r_N) \rangle = \langle \phi_1(r_1) \ldots \phi_N(r_N) \rangle \tag{41}$$

exists. The numbers x_j are called the scaling dimensions, and are related to the various critical exponents. They are related to the conformal weights $(h_j, \bar{h}_j)$ by $x_j = h_j + \bar{h}_j$; the difference $h_j - \bar{h}_j = s_j$ is called the spin of ϕ_j, and describes its behaviour under rotations. There are also *boundary operators*, localised on the boundary, which have only a single conformal weight equal to their scaling dimension.

The theory is developed independently of any particular set of fundamental fields or lagrangian. An important role in this is played by the stress tensor $T^{\mu\nu}(r)$, defined as the local response of the action to a change in the metric:

$$\delta S = (1/4\pi) \int_{\mathscr{D}} T^{\mu\nu} \delta g_{\mu\nu}\, d^2r. \tag{42}$$

Invariance under local rotations and scale transformations usually implies that $T^{\mu\nu}$ is symmetric and traceless: $T^{\mu}_{\mu} = 0$. This also implies invariance under conformal transformations, corresponding to $\delta g_{\mu\nu} \propto f(r) g_{\mu\nu}$.

In two-dimensional flat space, infinitesimal coordinate transformations $r^\mu \to r'^\mu = r^\mu + \alpha^\mu(r)$ correspond to infinitesimal transformations of the metric with $\delta g^{\mu\nu} = -(\partial^\mu \alpha^\mu + \partial^\nu \alpha^\nu)$. It is important to realise that under these transformations the underlying lattice, or UV cut-off, is not transformed. Otherwise they would amount to a trivial reparametrisation. For a conformal transformation, $\alpha^\mu(r)$ is given by an analytic function: in complex coordinates $(z, \bar{z})$, $\partial_{\bar{z}} \alpha^z = 0$, so $\alpha^z \equiv \alpha(z)$ is holomorphic. However, such a function cannot be small everywhere (unless it is constant), so it is necessary to consider coordinate transformations which are not everywhere conformal.

Consider therefore two concentric semicircles Γ_1 and Γ_2 in the upper half plane, centred on the origin, and of radii $R_1 < R_2$. For $|r| < R_1$ let α^μ be conformal, with $\alpha^z = \alpha(z)$, while for $|r| > R_2$ take $\alpha^\mu = 0$. In between, α^μ is not conformal, but is differentiable, so that $\delta S = (-1/2\pi)\int_{R_1<|r|<R_2} T^{\mu\nu}\partial_\mu \alpha_\nu\, d^2r$. This can be integrated by parts to give a term $(1/2\pi)\int_{R_1<|r|<R_2} \partial_\mu T^{\mu\nu} \alpha_\nu\, d^2r$ (which must vanish because α_ν is arbitrary

in this region, implying that $\partial_\mu T^{\mu\nu} = 0$) and two surface terms. That on Γ_2 vanishes because $\alpha^\mu = 0$ there. We are left with

$$\delta S = (1/2\pi) \int_{\Gamma_1} T^{\mu\nu} \alpha_\mu \epsilon^{\nu\lambda} \, d\ell_\lambda, \tag{43}$$

where $d\ell^\lambda$ is the line element along Γ_1.

The fact that $T^{\mu\nu}$ is conserved means, in complex coordinates, that $\partial_{\bar z} T_{zz} = \partial_z T_{\bar z \bar z} = 0$, so that the correlations functions of $T(z) \equiv T_{zz}$ are holomorphic functions of z, while those of $\overline{T} \equiv T_{\bar z \bar z}$ are antiholomorphic. Eq. (43) may then be written

$$\delta S = (1/2\pi i) \int_{\Gamma_1} T(z) \alpha(z) \, dz + \text{c.c.} \tag{44}$$

In any field theory with a boundary, it is necessary to impose some boundary condition. It can be argued that any translationally invariant boundary condition flows under the RG to conditions satisfying $T_{xy} = 0$, which in complex coordinates means that $T = \overline{T}$ on the real axis. This means that the correlators of $\overline{T}$ are those of T analytically continued into the lower half plane. The second term in (44) may then be dropped if the contour in the first term is around a complete circle.

The conclusion of all this is that the effect of an infinitesimal conformal transformation on any correlator of observables inside Γ_1 is the same as inserting a contour integral $\int T(z)\alpha(z)\, dz/2\pi i$ into the correlator.

Another important element of CFT is the operator product expansion (OPE) of the stress tensor with other local operators. Since T is holomorphic, this has the form

$$T(z) \cdot \phi(0) = \sum_n z^{-n-2} \phi^{(n)}(0), \tag{45}$$

where the $\phi^{(n)}$ are (possibly new) local operators. By taking $\alpha(z) \propto z$ (corresponding to a scale transformation) we see that $\phi^{(0)} = h\phi$, where h is its scaling dimension. Similarly, by taking $\alpha = \text{const.}$, $\phi^{(-1)} = \partial_x \phi$. Local operators for which $\phi^{(n)}$ vanishes for $n \geqslant 1$ are called primary. T itself is not primary: its OPE with itself takes the form

$$T(z) \cdot T(0) = c/2z^4 + (2/z^2) T(0) + (1/z) \partial_z T(0) + \cdots, \tag{46}$$

where c is the conformal anomaly number, ubiquitous in CFT. For example, the partition function on a long cylinder of length L and circumference ℓ behaves as $\exp(\pi c L/\ell)$, cf. (10).

5.2. Radial quantisation

This is the most important concept in understanding the link between SLE and CFT. We introduce it in the context of boundary CFT. As before, suppose there is some set of fundamental fields $\{\psi(r)\}$, with a Gibbs measure $e^{-S[\psi]}[d\psi]$. Let Γ be a semicircle in the upper half plane, centered on the origin. The Hilbert space of the BCFT is the function space (with a suitable norm) of field configurations $\{\psi_\Gamma\}$ on Γ.

The vacuum state is given by weighting each state $|\psi'_\Gamma\rangle$ by the (normalised) path integral restricted to the interior of Γ and conditioned to take the specified values ψ'_Γ on the boundary

$$|0\rangle = \int [\mathrm{d}\psi'_\Gamma] \int_{\psi_\Gamma=\psi'_\Gamma} [\mathrm{d}\psi] \mathrm{e}^{-S[\psi]} |\psi'_\Gamma\rangle. \tag{47}$$

Note that because of scale invariance different choices of the radius of Γ are equivalent, up to a normalisation factor.

Similarly, inserting a local operator $\phi(0)$ at the origin into the path integral defines a state $|\phi\rangle$. This is called the operator-state correspondence of CFT. If we also insert $(1/2\pi\mathrm{i})\int_C z^{n+1} T(z)\,\mathrm{d}z$, where C lies inside Γ, we get a state $L_n|\phi\rangle$. The L_n act linearly on the Hilbert space. From the OPE (45) we see that $L_n|\phi\rangle \propto |\phi^{(n)}\rangle$, and that, in particular, $L_0|\phi\rangle = h_\phi|\phi\rangle$. If ϕ is primary, $L_n|\phi\rangle = 0$ for $n \geqslant 1$. From the OPE (46) of T with itself follow the commutation relations for the L_n

$$[L_n, L_m] = (n-m)L_{n+m} + \tfrac{1}{12}cn(n^2-1)\delta_{n,-m}, \tag{48}$$

which are known as the Virasoro algebra. The state $|\phi\rangle$ together with all its descendants, formed by acting on $|\phi\rangle$ an arbitrary number of times with the L_n with $n \leqslant -1$, give a highest weight representation (where the weight is defined as the eigenvalue of $-L_0$).

There is another way of generating such a highest weight representation. Suppose the boundary conditions on the negative and positive real axes are both conformal, that is they satisfy $T = \overline{T}$, but they are different. The vacuum with these boundary conditions gives a highest weight state which it is sometimes useful to think of as corresponding to the insertion of a 'boundary condition changing' (bcc) operator at the origin. An example is the continuum limit of an Ising model in which the spins on the negative real axis are -1, and those on the positive axis are $+1$.

5.3. Curves and states

In this section, we describe a way of associating states in the Hilbert space of the BCFT with the growing curves of the Loewner process. This was first understood by Bauer and Bernard [10], but we shall present the arguments slightly differently.

The boundary conditions associated with a bcc operator guarantee the existence, on the lattice, of a domain wall connecting the origin to infinity. Given a particular realisation γ, we can condition the Ising spins on its existence. We would like to be able to assume that this property continues to hold in the continuum limit: that is, we can condition the fields $\{\psi\}$ on the existence of a such a curve. However, this involves conditioning on an event with probability zero: it turns out that in general the probability that, with respect to the measure in the path integral, the probability that a domain wall hits the real axis somewhere in an interval of length ϵ vanishes like ϵ^h. In what follows we shall regard ϵ as small but fixed, and assume that the usual properties of SLE are applicable to this more general case.

Any such curve may be generated by a Loewner process: denote as before the part of the curve up to time t by γ_t. The existence of this curve depends on only the field configurations ψ in the interior of Γ, as long as γ_t lies wholly inside this region. Then we can condition the fields contributing to the path integral on the existence of γ_t, thus defining a state

$$|\gamma_t\rangle = \int [\mathrm{d}\psi'_\Gamma] \int_{\psi_\Gamma=\psi'_\Gamma;\gamma_t} [\mathrm{d}\psi] \mathrm{e}^{-S[\psi]} |\psi'_\Gamma\rangle. \tag{49}$$

The path integral (over the whole of the upper half plane, not just the interior of Γ), when conditioned on γ_t, gives a measure $\mathrm{d}\mu(\gamma_t)$ on these curves. The state

$$|h\rangle = |h_t\rangle \equiv \int \mathrm{d}\mu(\gamma_t)|\gamma_t\rangle \tag{50}$$

is in fact independent of t, since it is just given by the path integral conditioned on there being a curve connecting the origin to infinity, as guaranteed by the boundary conditions. In fact we see that $|h\rangle$ is just the state corresponding to a boundary condition changing operator at the origin.

However, $\mathrm{d}\mu(\gamma_t)$ is also given by the measure on a_t in Loewner evolution, through the iterated sequence of conformal mappings satisfying $\mathrm{d}\hat{g}_t = 2\mathrm{d}t/\hat{g}_t - \mathrm{d}a_t$. This corresponds to an infinitesimal conformal mapping of the upper half plane minus K_t. As explained in the previous section, $\mathrm{d}\hat{g}_t$ corresponds to inserting $(1/2\pi\mathrm{i})\int_C (2\mathrm{d}t/z - \mathrm{d}a_t)T(z)\,\mathrm{d}z$. In operator language, this corresponds to acting on $|\gamma_t\rangle$ with $2L_{-2}\mathrm{d}t - L_{-1}\mathrm{d}a_t$ where $L_n = (1/2\pi\mathrm{i})\int_C z^{n+1}T(z)\,\mathrm{d}z$. Thus, for any $t_1 < t$

$$|g_{t_1}(\gamma_t)\rangle = \mathbf{T}\exp\left(\int_0^{t_1} (2L_{-2}\,\mathrm{d}t' - L_{-1}\mathrm{d}a_{t'})\right)|\gamma_t\rangle, \tag{51}$$

where $\mathbf{T}$ denotes a time-ordered exponential.

The measure on γ_t is the product of the measure of $\gamma_t \setminus \gamma_{t_1}$, conditioned on γ_{t_1}, with the unconditioned measure on γ_{t_1}. The first is the same as the unconditioned measure on $g_{t_1}(\gamma_t)$, and the second is given by the measure on $a_{t'}$ for $t' \in [0, t_1]$. Thus, we can rewrite both the measure and the state in (50) as

$$|h_t\rangle = \int \mathrm{d}\mu(g_{t_1}(\gamma_t)) \int \mathrm{d}\mu(a_{t';t'\in[0,t_1]})\mathbf{T}\exp\left(\int_{t_1}^{0} (2L_{-2}\,\mathrm{d}t' - L_{-1}\,\mathrm{d}a_{t'})\right)|g_{t_1}(\gamma_t)\rangle. \tag{52}$$

For SLE, a_t is proportional to a Brownian process. The integration over realisations of this for $t' \in [0, t_1]$ may be performed by breaking up the time interval into small segments of size δt, expanding out the exponential to $\mathrm{O}(\delta t)$, using $(B_{\delta t})^2 \approx \delta t$, and re-exponentiating. The result is

$$|h_t\rangle = \exp(-(2L_{-2} - (\kappa/2)L_{-1}^2)t_1)|h_{t-t_1}\rangle. \tag{53}$$

But, as we argued earlier, $|h_t\rangle$ is independent of t, and therefore

$$(2L_{-2} - (\kappa/2)L_{-1}^2)|h\rangle = 0. \tag{54}$$

This means that the descendant states $L_{-2}|h\rangle$ and $L_{-1}^2|h\rangle$ are linearly dependent. We say that the Virasoro representation corresponding to $|h\rangle$ has a null state at level 2.

From this follow an number of important consequences. Acting on (54) with L_1 and L_2, and using the Virasoro algebra (48) and the fact that $L_1|h\rangle = L_2|h\rangle = 0$ while $L_0|h\rangle = h|h\rangle$, leads to:

$$h = h_{2,1} = \frac{6-\kappa}{2\kappa}; \tag{55}$$

$$c = \frac{(3\kappa-8)(6-\kappa)}{2\kappa}. \tag{56}$$

These are the fundamental relations between the parameter κ of SLE and the data of CFT. The conventional notation $h_{2,1}$ comes from the Kac formula in CFT which we do not need here. In fact this is appropriate to the case $\kappa < 4$: for $\kappa > 4$ it corresponds to $h_{1,2}$. (To further confuse the matter, some authors reverse the labels.) Note that the boundary exponent h parametrises the failure of locality in (23). From CFT we may also deduce that, with respect to the path integral measure, the probability that a curve connects small intervals of size ϵ about points r_1, r_2 on the real axis behaves like

$$\epsilon^{2h_{2,1}}\langle\phi_{2,1}(r_1)\phi_{2,1}(r_2)\rangle \propto \left(\frac{\epsilon}{|r_1-r_2|}\right)^{2h_{2,1}}. \tag{57}$$

Such a result, elementary in CFT, is difficult to obtain directly from SLE in which the curves are conditioned to begin and end at *given* points.

Note that the central charge c vanishes when either locality ($\kappa = 6$) or restriction ($\kappa = 8/3$) hold. These cases correspond to the continuum limit of percolation and self-avoiding walks, respectively, corresponding to formal limits $Q \to 1$ in the Potts model and $n \to 0$ in the $\mathrm{O}(n)$ model for which the unconditioned partition function is trivial.

5.4. Differential equations

In this section, we discuss how the linear second order differential equations for various observables which arise from the stochastic aspect of SLE follow equivalently from the null state condition in CFT. In this context they are known as the BPZ equations [33]. As an example consider Schramm's formula (30) for the probability P that a point ζ lies to the right of γ, or equivalently the expectation value of the indicator function $\mathcal{O}(\zeta)$ which is 1 if this is satisfied and zero otherwise. In SLE, this expectation value is with respect to the measure on curves which connect the point a_0 to infinity. In CFT, as explained above, we can only consider curves which intersect some ϵ-neighbourhood on the real axis. Therefore P should be written as a ratio of expectation values with respect to the CFT measure

$$P(\zeta;a_0) = \lim_{r_2\to\infty} \frac{\langle\phi_{2,1}(a_0)\mathcal{O}(\zeta)\phi_{2,1}(r_2)\rangle}{\langle\phi_{2,1}(a_0)\phi_{2,1}(r_2)\rangle}. \tag{58}$$

We can derive differential equations for the correlators in the numerator and denominator by inserting into each of them a factor $(1/(2\pi\mathrm{i})\int_\Gamma \alpha(z)T(z)\,\mathrm{d}z + \text{c.c.}$, where $\alpha(z) = 2/(z-a_0)$, and Γ is a small semicircle surrounding a_0. This is equivalent to making the infinitesimal transformation $z \to z + 2\epsilon/(z - a_0)$. As before, the c.c. term is equivalent to extending the contour in the first term to a full circle. The effect of this insertion may be evaluated in two ways: by shrinking the contour onto a_0 and using the OPE between T and $\phi_{2,1}$ we get

$$2L_{-2}\phi_{2,1}(a_0) = (\kappa/2)L_{-1}^2\phi_{2,1}(a_0) = \partial_{a_0}^2\phi_{2,1}(a_0), \tag{59}$$

while wrapping it about ζ (in a clockwise sense) we get

$$-(2/(\zeta - a_0))\partial_\zeta \mathcal{O} - \big(2/(\bar{\zeta} - a_0)\big)\partial_{\bar{\zeta}}\mathcal{O}. \tag{60}$$

The effect on $\phi_{2,1}(r_2)$ vanishes in the limit $r_2 \to \infty$. As a result we can ignore the variation of the denominator in this case. Equating (59) and (60) inside the correlation function in the numerator then leads to the differential equation (29) for P found in Section 4.1.

5.4.1. Calogero–Sutherland model

While many of the results of SLE may be re-derived in CFT with less rigour but perhaps greater simplicity, the latter contains much more information which is not immediately apparent from the SLE perspective. For example, one may consider correlation functions $\langle\phi_{1,2}(r_1)\phi_{1,2}(r_2)\cdots\phi_{2,1}(r_N)\cdots\rangle$ of multiple boundary condition changing operators with other operators either in the bulk or on the boundary. Evaluating the effect of an insertion $(1/2\pi i)\int_\Gamma T(z)\,\mathrm{d}z/(z - r_j)$ where Γ surrounds r_j leads to a second order differential equation satisfied by the correlation function for *each j*.

This property is very powerful in the radial version. Consider the correlation function

$$C_\Phi(\theta_1, \ldots, \theta_N) = \langle\phi_{2,1}(\theta_1)\cdots\phi_{2,1}(\theta_N)\Phi(0)\rangle \tag{61}$$

of $N\phi_{2,1}$ operators on the boundary of the unit disc with a single bulk operator Φ at the origin. Consider the effect of inserting $(1/2\pi\mathrm{i})\int_\Gamma \alpha_j(z)T(z)\,\mathrm{d}z$, where (cf. (26))

$$\alpha_j(z) = -z\frac{z + \mathrm{e}^{\mathrm{i}\theta_j}}{z - \mathrm{e}^{\mathrm{i}\theta_j}} \tag{62}$$

and Γ surrounds the origin. Once again, this may be evaluated in two ways: by taking Γ up to the boundary, with exception of small semicircles around the points $\mathrm{e}^{\mathrm{i}\theta_k}$, we get G_jC_Φ, where G_j is the second order differential operator

$$G_j = -\frac{\kappa}{2}\frac{\partial^2}{\partial\theta_j^2} + \frac{h_{2,1}}{6} + \frac{c}{12} - \sum_{k\neq j}\left(\cot\frac{\theta_k - \theta_j}{2}\frac{\partial}{\partial\theta_k} - \frac{1}{2\sin^2(\theta_k - \theta_j)/2}h_{2,1}\right). \tag{63}$$

The first three terms come from evaluating the contour integral near $\mathrm{e}^{\mathrm{i}\theta_j}$, where α_j acts like $2L_{-2} - \frac{1}{6}L_0 - \frac{c}{12}$ (the term $\frac{c}{12}$ comes from the curvature of the boundary), and the term with $k \neq j$ from the contour near $\mathrm{e}^{\mathrm{i}\theta_k}$, where it acts like $\alpha_j(\mathrm{e}^{\mathrm{i}\theta_k})L_{-1} + \mathrm{Re}\,\alpha_j'(\mathrm{e}^{\mathrm{i}\theta_k})L_0$.

On the other hand, shrinking the contour down on the origin we see that $\alpha_j(z) = z + \mathrm{O}(z^2)$, so that on $\Phi(0)$ it has the effect of $L_0 + \bar{L}_0 + \cdots$, where the omitted terms involve the L_n and $\overline{L}_n$ with $n > 0$. Assuming that Φ is primary, these other terms vanish, leaving simply $(L_0 + \overline{L}_0)\Phi = x_\Phi \Phi$. Equating the two evaluations we find the differential equation

$$G_j C_\Phi = x_\Phi C_\Phi. \tag{64}$$

In general there is an $(N-1)$-dimensional space of independent differential operators G_j with a common eigenfunction C_Φ. (There is one fewer dimension because they all commute with the total angular momentum $\sum_j(\partial/\partial\theta_j)$.) For the case $N = 2$, setting $\theta = \theta_2 - \theta_1$, we recognise the differential operator in Section 4.3.3.

In general these operators are not self-adjoint and their spectrum is difficult to analyse. However, if we form the equally weighted linear combination $G \equiv \sum_{j=1}^N G_j$, the terms with a single derivative may be written in the form $\sum_k(\partial V/\partial\theta_k)(\partial/\partial\theta_k)$ where V is a potential function. In this case it is well known from the theory of the Fokker–Plank equation that G is related by a similarity transformation to a self-adjoint operator. In fact [36] if we form $|\Psi_N|^{2/\kappa} G |\Psi_N|^{-2/\kappa}$, where $\Psi_N = \prod_{j<k}(\mathrm{e}^{\mathrm{i}\theta_j} - \mathrm{e}^{\theta_k})$ is the 'free-fermion' wave function on the circle, the result is, up to calculable constants the well-known N-particle Calogero–Sutherland hamiltonian

$$H_N(\beta) = -\frac{1}{2}\sum_{j=1}^N \frac{\partial^2}{\partial\theta_j^2} + \frac{\beta(\beta-2)}{16}\sum_{j<k}\frac{1}{\sin^2(\theta_j - \theta_k)/2} \tag{65}$$

with $\beta = 8/\kappa$. It follows that the scaling dimensions of bulk operators like Φ are simply related to eigenvalues Λ_N of H_N by

$$x_\Phi = (\kappa/N)\Lambda_N(8/\kappa) - (4/\kappa N)E_N^{\mathrm{ff}} + \tfrac{1}{6}h_{2,1} + \tfrac{1}{12}c, \tag{66}$$

where $E_N^{\mathrm{ff}} = \frac{1}{24}N(N^2-1)$. Similarly C_Φ is proportional to the corresponding eigenfunction. In fact the ground state (with conventional boundary conditions) turns out to correspond to the bulk N-leg operator discussed in Section 2.4.2. The corresponding correlator is $|\Psi_N|^{2/\kappa}$.

6. Related ideas

6.1. Multiple SLEs

We pointed out earlier that the boundary operators $\phi_{2,1}$ correspond to the continuum limits of lattice curves which hit the boundary at a given point. For a single curve, these are described by SLE, and we have shown in that case how the resulting differential equations also appear in CFT. Using the N-particle generalisation of the CFT results of the previous section, we may now 'reverse engineer' the problem and conjecture the generalisation of SLE to N curves.

The expectation value of some observable $\mathcal{O}$ given that N curves, starting at the origin, hit the boundary at $(\theta_1, \ldots, \theta_N)$ is

$$P_{\mathcal{O}}(\theta_1,\ldots,\theta_N)=\frac{F_{\mathcal{O}}(\theta_1,\ldots,\theta_N)}{F_{\mathbf{1}}(\theta_1,\ldots,\theta_N)}, \tag{67}$$

where $F_{\mathcal{O}}=\langle\mathcal{O}\phi_{2,1}(\mathrm{e}^{\mathrm{i}\theta_1})\ldots\phi_{2,1}(\mathrm{e}^{\mathrm{i}\theta_N})\Phi_N(0)\rangle$. This satisfies the BPZ equation

$$G_jF_{\mathcal{O}}=\langle(\delta_j\mathcal{O}\phi_{2,1}(\mathrm{e}^{\mathrm{i}\theta_1})\ldots\phi_{2,1}(\mathrm{e}^{\mathrm{i}\theta_N})\Phi_N(0)\rangle, \tag{68}$$

where $\delta_j\mathcal{O}$ is the variation in $\mathcal{O}$ under α_j. If we now write $F_{\mathcal{O}}=F_{\mathbf{1}}\cdot P_{\mathcal{O}}$ and use the fact that $G_jF_{\mathbf{1}}=x_{\Phi}F_{\mathbf{1}}$, we find a relatively simple differential equation for $P_{\mathcal{O}}$, since the non-derivative terms in G_j cancel. There is also a complication since the second derivative gives a cross term proportional to $(\partial_{\theta_j}F_{\mathbf{1}})(\partial_{\theta_j}P_{\mathcal{O}})$. However, this may be evaluated from the explicit form $F_{\mathbf{1}}=|\Psi_N|^{2/\kappa}$. The result is

$$\left(\frac{\kappa}{2}\frac{\partial^2}{\partial\theta_j^2}+\sum_{k\neq j}\cot\frac{\theta_k-\theta_j}{2}\left(\frac{\partial}{\partial\theta_k}-\frac{\partial}{\partial\theta_j}\right)\right)P_{\mathcal{O}}=\delta_jP_{\mathcal{O}}, \tag{69}$$

where the right-hand side comes from the variation in $\mathcal{O}$.

The left-hand side may be recognised as the generator (the adjoint of the Fokker–Planck operator) for the stochastic process:

$$\mathrm{d}\theta_j=\sqrt{\kappa}\mathrm{d}B_t+\sum_{k/2\neq j}\rho_{k/2}\cot\big((\theta_j-\theta_k)/2\big)\,\mathrm{d}t; \tag{70}$$

$$\mathrm{d}\theta_k=\cot((\theta_k-\theta_j)/2)\,\mathrm{d}t, \tag{71}$$

where $\rho_k=2$. [For general values of the parameters ρ_k this process is known as (radial) SLE$(\kappa,\vec{\rho})$, although this is more usually considered in the chordal version. It has been argued [37] that this applies to the level lines of a free gaussian field with piecewise constant Dirichlet boundary conditions: the parameters ρ_k are related to the size of the discontinuities at the points $\mathrm{e}^{\mathrm{i}\theta_k}$. SLE$(\kappa,\vec{\rho})$ has also been used to give examples of restriction measures on curves which are not reflection symmetric [38].]

We see that $\mathrm{e}^{\mathrm{i}\theta_j}$ undergoes Brownian motion but is also repelled by the other particles at $\mathrm{e}^{\mathrm{i}\theta_k}(k\neq j)$: these particles are themselves repelled deterministically from $\mathrm{e}^{\mathrm{i}\theta_j}$. The infinitesimal transformation α_j corresponds to the radial Loewner equation

$$\frac{\mathrm{d}g_{j,t}}{\mathrm{d}t}=-g_{j,t}\frac{g_{j,t}+\mathrm{e}^{\mathrm{i}\theta_{j,t}}}{g_{j,t}-\mathrm{e}^{\mathrm{i}\theta_{j,t}}}. \tag{72}$$

The conjectured interpretation of this is as follows: we have N non-intersecting curves connecting the boundary points $\mathrm{e}^{\mathrm{i}\theta_{k,0}}$ to the origin. The evolution of the jth curve in the presence of the others is given by the radial Loewner equation with, however, the driving term not being simple Brownian motion but instead the more complicated process (70) and (71).

However, from the CFT point of view we may equally well consider the linear combination $\sum_jG_j$. The Loewner equation is now

$$\dot{g}_t=-g_t\sum_{j=1}^{N}\frac{g_t+\mathrm{e}^{\mathrm{i}\theta_{j,t}}}{g_t-\mathrm{e}^{\mathrm{i}\theta_{j,t}}}, \tag{73}$$

where

$$d\theta_j = \sqrt{\kappa}dB_t^j + 2\sum_{k\neq j}\cot((\theta_j - \theta_k)/2)\,dt. \tag{74}$$

This is known in the theory of random matrices as Dyson's Brownian motion. It describes the statistics of the eigenvalues of unitary matrices. The conjectured interpretation is now in terms of N random curves which are all growing in each other's mutual presence at the *same* mean rate (measured in Loewner time). From the point of view of SLE, it is by no means obvious that the measure on N curves generated by process (70)–(72) is the same as that given by (73) and (74). However, CFT suggests that, for curves which are the continuum limit of suitable lattice models, this is indeed the case.

6.2. Other variants of SLE

So far we have discussed only chordal SLE, which describes curves connecting distinct points on the boundary of a simple connected domain, and radial SLE, in which the curve connects a boundary point to an interior point. Another simple variant is dipolar SLE [39], in which the curve is constrained to start at boundary point and to end on some finite segment of the boundary not containing the point. The canonical domain is an infinitely long strip, with the curve starting a point on one edge and ending on the other edge. This set-up allows the computation of several interesting physical quantities.

The study of SLE in multiply connected domains is very interesting. Their conformal classes are characterised by a set of moduli, which change as the curve grows. Friedrich and Kalkkinen [40] have argued that SLE in such a domain is characterised by diffusion in moduli space as well as diffusion on the boundary.

It is possible to rewrite the differential equations which arise from null state conditions in extended CFTs (for example super-conformal CFTs [41] and WZWN models [42]) in terms of the generators of stochastic conformal mappings which generalise that of Loewner. However, a physical interpretation in terms of the continuum limit of lattice curves appears so far to be missing.

6.3. Other growth models

SLE is in fact just one very special, solvable, example of an approach to growth processes in two dimensions using conformal mappings which has been around for a number of years. For a recent review see [43]. The prototypical problem of this type is diffusion-limited aggregation (DLA). In this model of cluster formation, particles of finite radius diffuse in, one by one, from infinity until they hit the existing cluster, where they stick. The probability of sticking at a given point is proportional to the local electric field, if we imagine the cluster as being charged. The resultant highly branched structures are very similar to those observed in smoke particles, and in viscous fingering experiments where one fluid is forced into another in which it is immiscible. Hastings and Levitov [44] proposed an approach to this problem using conformal mappings. At each time t, the boundary of the cluster is described by the conformal mapping $f_t(z)$ which takes it to the unit disc. The cluster is grown by adding a small semicircular piece to the boundary. The way this changes f_t is well known according to a theorem of Hadamard. The difficulty is that the probability of adding this piece at a given point depends on the local electric field which itself depends on f'_t. The equation of motion for f_t is therefore more complicated than in SLE. Moreover, it may be shown that almost all initially smooth boundary curves evolve towards a finite-time singularity: this is thought to be responsible for branching, but just at this point the equations must be regularised to reflect the finite size of the particles (or, in viscous fingering, the effects of finite surface tension.)

It is also possible to generate branching structures by making the driving term a_t in Loewner's equation discontinuous, for example taking it to be a Levy process. Unfortunately this does not appear to describe a physically interesting model.

Finally, Hastings [45] has proposed two related growth models which each lead, in the continuum limit, to SLE. These are very similar to DLA, except that growth is only allowed at the tip. The first, called the arbitrary Laplacian random walk, takes place on the lattice. The tip moves to one of the neighbouring unoccupied sites r with relative probability $E(r)^\eta$, where $E(r)$ is the lattice electric field, that is the potential difference between the tip and r, and η is a parameter. The second growth model takes place in the continuum via iterated conformal mappings, in which pieces of length ℓ_1 are added to the tip, but shifted to the left or right relative to the previous growth direction by a random amount $\pm\ell_2$. This model depends on the ratio ℓ_2/ℓ_1, and leads, in the continuum limit, to SLE_κ with $\kappa = \ell_2/4\ell_1$. For the lattice model there is no universal relation between κ and η, except for $\eta = 1$, which is the same as the loop-erased random walk (Section 2.2) and converges to SLE_2.

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